Gauge fixing and classical dynamical r-matrices in ISO(2, 1)-Chern-Simons theory

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March 23, 2012

Abstract

We apply the Dirac gauge fixing procedure to Chern-Simons theory with gauge group $\mathrm{ISO}(2,1)$ on manifolds $\mathbb{R}\times S$, where S is a punctured oriented surface of general genus. For all gauge fixing conditions that satisfy certain structural requirements, this yields an explicit description of the Poisson structure on the moduli space of flat $\mathrm{ISO}(2,1)$-connections on S in terms of classical dynamical r-matrices for $\mathfrak{iso}(2,1)$. We show that the Poisson structures and classical dynamical r-matrices arising from different gauge fixing conditions are related by dynamical $\mathrm{ISO}(2,1)$-valued transformations that generalise the usual gauge transformations of classical dynamical r-matrices. By means of these transformations, it is possible to classify all Poisson structures and classical dynamical r-matrices obtained from such gauge fixings. Generically these Poisson structures combine classical dynamical r-matrices for non-conjugate Cartan subalgebras of $\mathfrak{iso}(2,1)$.

1 Introduction

Moduli spaces of flat connections on punctured Riemann surfaces and their quantisation are of interest to both mathematics and physics due to their rich mathematical structure and their links with a variety of topics from geometry, algebra and gauge theory. From the physics perspective, a major motivation for their study is their role in Chern-Simons theory. Moduli spaces of flat connections can be viewed as the gauge-invariant or reduced phase spaces of Chern-Simons theories. Their quantisation is thus related to structures arising in the quantisation of Chern-Simons theory such as quantum groups, aspects of knot theory [37] and topological quantum field theories. Another important feature of the theory is its relation to three-dimensional gravity [1, 36, 38].

The quantisation of moduli spaces of flat G-connections and their relation to quantum Chern-Simons theory are well understood for the case of compact, semisimple Lie groups G. In this setting, quantisation can be achieved via many formalisms, and most of these formalisms involve the representation theory of a quantum group, namely the q-deformed universal enveloping algebra $U_q(\mathfrak{g})$ at a root of unity. In the case of non-compact non-semisimple Lie groups, the quantisation proves more difficult. Although there are partial results on the

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quantisation of these cases via analytic continuation [40] and results for specific Lie groups [39, 11, 30, 27, 26], there is currently no general framework to address this case.

From the viewpoint of Hamiltonian quantisation formalisms, these difficulties are related to the fact that Chern-Simons theory and the associated moduli spaces of flat connections can be viewed as constrained Hamiltonian systems. In the non-compact setting, representation-theoretical complications lead to difficulties in the implementation of the constraint operators in the quantum theory. It therefore seems advisable to also consider other approaches to the quantisation of moduli spaces of flat connections with non-compact gauge groups. This includes in particular "quantisation after constraint implementation" approaches, which proceed by first applying the Dirac gauge fixing formalism to the classical theory and then quantising the resulting Poisson structure. However, besides partial results for $\mathrm{SL}(2,\mathbb{C})$-Chern-Simons theory [10, 13, 32], this avenue has not been pursued yet.

An independent mathematical motivation for investing gauge fixing procedures related to moduli spaces of flat connections arises from Poisson geometry. Such gauge fixing procedures can be interpreted as the Poisson counterpart of symplectic reduction. Moduli spaces of flat connections played an important role in many interesting developments in this subject such as Lie-group-valued moment maps, see [2] and the references therein. Moreover, the symplectic structure on the moduli space can be characterised in terms of certain Poisson structures from the theory of Poisson-Lie groups. Gauge fixing in this context has been shown to give rise to classical dynamical r-matrices in some cases [21, 20].

In this article, we undertake a systematic investigation of Dirac gauge fixing for the moduli space of flat $\mathrm{ISO}(2,1)$-connections on a Riemann surface $S_{g,n}$ of general genus g and with $n \geq 2$ punctures. Our choice of the group $\mathrm{ISO}(2,1) = \mathrm{SO}_+(2,1) \ltimes \mathbb{R}^3$ is motivated by the fact that it is an example of a non-compact non-semisimple Lie group and that Chern-Simons theory with gauge group $\mathrm{ISO}(2,1)$ is closely related to (2+1)-gravity [1, 36].

Our starting point is a description of the moduli space of flat $\mathrm{ISO}(2,1)$-connections on $S_{g,n}$ in terms of a Poisson structure on the direct product $\mathcal{P}_{\mathrm{ext}} = \mathrm{ISO}(2,1)^{n+2g}$ due to Alekseev and Malkin [5] and Fock and Rosly [22]. It is shown in [5, 22] that this Poisson structure is given in terms of certain Poisson structures related to Poisson-Lie groups and after reduction induces the canonical Poisson structure on the moduli space. In the case of the group $\mathrm{ISO}(2,1)$, this Poisson structure is given by the direct product of n copies of the dual Poisson-Lie structure on $\mathrm{ISO}(2,1)$ and $2g$ copies of the cotangent bundle Poisson structure for $\mathrm{SO}_+(2,1)$ [30]:

$$\mathcal{P}_{\mathrm{ext}} = \underbrace{\mathrm{ISO}(2,1)^* \times \ldots \mathrm{ISO}(2,1)^*}_{n\,\times} \times \underbrace{T^*(\mathrm{SO}_+(2,1)) \times \ldots \times T^*(\mathrm{SO}_+(2,1))}_{2g\,\times}.$$

From the physics perspective, the Poisson manifold $(\mathcal{P}_{\mathrm{ext}}, \{\ ,\ \})$ can be viewed as a constrained system with a set of six first-class constraints from which the moduli space and its symplectic structure are obtained after constraint implementation. This allows one to choose appropriate gauge fixing conditions and to apply the Dirac gauge fixing procedure [15, 16] to this Poisson structure. We explicitly compute the resulting Poisson bracket for a large class of gauge fixing conditions and investigate the resulting Poisson structures. This yields our first central result:

Theorem. *For suitable gauge fixing conditions, the Dirac gauge fixing procedure applied to* $(\mathcal{P}_{\mathrm{ext}}, \{\ ,\ \})$ *gives rise to a Poisson structure on* $\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$ *which is determined uniquely by a solution of the classical dynamical Yang-Baxter equation.*

In particular, we find that this Poisson structure on $\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$ is given by a formula directly analogous to the original Poisson structure on $\mathcal{P}_{\mathrm{ext}}$. The only difference is

that the classical r-matrix in the original definition is replaced by a solution of the classical dynamical Yang-Baxter equation for $\mathfrak{iso}(2,1)$ whose two dynamical variables parametrise $\mathbb{R}^2 \subset \mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$.

We then investigate the relation between the Poisson structures and solutions of the classical dynamical Yang-Baxter equation that result from different choices of gauge fixing conditions. This leads to our second central result:

Theorem. *All solutions of the classical dynamical Yang-Baxter equation obtained from gauge fixing are related by dynamical* $\mathrm{ISO}(2,1)$*-valued transformations that generalise the gauge transformations of classical dynamical* r*-matrices from [19]. All such solutions are locally equivalent to one of two standard solutions corresponding to Cartan subalgebras of* $\mathfrak{iso}(2,1)$*.*

The second statement of the theorem refers to an interesting feature of our solutions that does not appear to have been observed in the literature yet. The solutions of the classical dynamical Yang-Baxter equation that arise from generic gauge fixing conditions are not associated with a fixed Cartan subalgebra of $\mathfrak{iso}(2,1)$ but combine classical dynamical r-matrices for two non-conjugate Cartan subalgebras of $\mathfrak{iso}(2,1)$.

Our paper is structured as follows. Section 2 contains the basic definitions, notation and conventions used in the remainder of the paper. Section 3 summarises the physics background and motivation of this work. It can be skipped without loss by the reader unfamiliar with this or interested mainly in the mathematical results. It contains a brief discussion of Chern-Simons theory on manifolds of topology $\mathbb{R} \times S_{g,n}$ and of the moduli space of flat connections on $S_{g,n}$ as a reduced or gauge-invariant phase space of Chern-Simons theory. It then explains the Dirac gauge fixing formalism and its relation to symplectic reduction and discusses the constraints and gauge fixing conditions imposed to obtain the moduli space.

Section 4 contains the first central result of this article, namely the explicit description of the Poisson structure resulting from Dirac gauge fixing for a general set of gauge fixing conditions. We show that the resulting Poisson structures are associated with solutions of the classical dynamical Yang-Baxter equation and discuss examples arising from specific choices of gauge fixing conditions as well as two simple standard solutions associated with Cartan subalgebras of $\mathfrak{iso}(2,1)$.

In Section 5 we introduce dynamical $\mathrm{ISO}(2,1)$-transformations which can be viewed as transitions between different gauge fixing conditions. We determine the associated transformations of the Dirac bracket and show how they can be interpreted as transitions between different solutions of the classical dynamical Yang-Baxter equation. In that sense, the dynamical $\mathrm{ISO}(2,1)$-transformations generalise the gauge transformations of classical dynamical r-matrices in [19]. We then apply these dynamical transformations to give a complete classification of the classical dynamical r-matrices and Poisson structures obtained from gauge fixing. Section 6 contains the outlook and our conclusions.

2 Notations and conventions

We denote by $\boldsymbol{e}_0 = (1,0,0)$, $\boldsymbol{e}_1 = (0,1,0)$, $\boldsymbol{e}_2 = (0,0,1)$ the standard basis of $\mathbb{R}^3$ and use Einstein's summation convention throughout this paper. Unless stated otherwise, all indices run from 0 to 2 and are raised and lowered with the three-dimensional Minkowski metric $\eta = \mathrm{diag}(1,-1,-1)$. By ε_{abc} we denote the totally antisymmetric tensor in three

dimensions with the convention $\varepsilon_{012} = 1$. For vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$, we use the notation $\eta(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y} = \eta_{ab} x^a y^b$ and $\boldsymbol{x}^2 = \boldsymbol{x} \cdot \boldsymbol{x}$, and we write $\boldsymbol{x} \wedge \boldsymbol{y}$ for the vector with components $(\boldsymbol{x} \wedge \boldsymbol{y})^a = \varepsilon^{abc} x_b y_c$. Note that this is a Lorentzian version of the wedge product which does not coincide with the standard one.

We denote by $\mathrm{SO}_+(2,1) \cong \mathrm{PSL}(2,\mathbb{R})$ the proper orthochronous Lorentz group in three dimensions and by $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$ its Lie algebra. In the following, we fix a set of generators $\{J_a\}_{a=0,1,2}$ of $\mathfrak{so}(2,1)$ such that the Lie bracket takes the form $[J_a, J_b] = \varepsilon_{ab}{}^c J_c$. As the representation of $\mathrm{PSL}(2,\mathbb{R})$ by $\mathrm{SO}_+(2,1)$ matrices coincides with its adjoint representation, we denote both representations by Ad in the following:

$$g \cdot J_a \cdot g^{-1} = \mathrm{Ad}(g)^b{}_a J_b \qquad \forall g \in \mathrm{SO}_+(2,1).$$

The Poincaré group in three dimensions is the semidirect product $\mathrm{ISO}(2,1) \equiv \mathrm{SO}_+(2,1) \ltimes \mathbb{R}^3$ of the proper orthochronous Lorentz group $\mathrm{SO}_+(2,1)$ and the translation group $\mathbb{R}^3$. We parametrise elements of $\mathrm{ISO}(2,1)$ as

$$(u, \boldsymbol{a}) = (u, 0) \cdot (\mathbb{1}, -\boldsymbol{j}) = (u, -\mathrm{Ad}(u)\boldsymbol{j}) \quad \text{with } u \in \mathrm{SO}_+(2,1), \boldsymbol{j}, \boldsymbol{a} \in \mathbb{R}^3.$$

The group multiplication law then takes the form

$$(u_1, \boldsymbol{a}_1) \cdot (u_2, \boldsymbol{a}_2) = (u_1 u_2, \boldsymbol{a}_1 + \mathrm{Ad}(u_1)\boldsymbol{a}_2).$$

A basis of the Lie algebra $\mathfrak{iso}(2,1)$ is given by the basis $\{J_a\}_{a=0,1,2}$ of $\mathfrak{so}(2,1)$ together with a basis $\{P_a\}_{a=0,1,2}$ of the abelian Lie algebra $\mathbb{R}^3$. In this basis, the Lie bracket takes the form

$$[J_a, J_b] = \varepsilon_{ab}{}^c J_c, \qquad [J_a, P_b] = \varepsilon_{ab}{}^c P_c, \qquad [P_a, P_b] = 0, \tag{1}$$

and a non-degenerate Ad-invariant symmetric bilinear form on $\mathfrak{iso}(2,1)$ is given by

$$\langle J_a, J_b \rangle = \langle P_a, P_b \rangle = 0, \qquad \langle J_a, P_b \rangle = \eta_{ab}. \tag{2}$$

All Cartan subalgebras of $\mathfrak{iso}(2,1)$ are abelian and can be parametrised in terms of two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ with $\boldsymbol{x}^2 \in \{1, -1\}$ and $\boldsymbol{x} \cdot \boldsymbol{y} = 0$ as

$$\mathfrak{h} = \mathrm{span}\{x^a P_a,\, x^a J_a + y^a P_a\} \tag{3}$$

If the vector $\boldsymbol{x}$ is timelike ($\boldsymbol{x}^2 = 1$), then the associated Cartan subalgebra $\mathfrak{h}$ is conjugate under the adjoint action of $\mathrm{ISO}(2,1)$ to $\mathrm{span}\{P_0, J_0\}$. If $\boldsymbol{x}$ is spacelike ($\boldsymbol{x}^2 = -1$), then $\mathfrak{h}$ is conjugate to $\mathrm{span}\{P_1, J_1\}$. Note that the set (3) with a lightlike vector $\boldsymbol{x} \in \mathbb{R}^3$ ($\boldsymbol{x}^2 = 0$) does not form a Cartan subalgebra of $\mathfrak{iso}(2,1)$ because it is not self-normalising.

In the following, we will also need the right- and left-invariant vector fields on $\mathrm{ISO}(2,1)$ associated with a basis $\{T_a\}_{a=0,\ldots,5}$ of $\mathfrak{iso}(2,1)$. They are given by

$$L_a f(h) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(e^{-tT_a} \cdot h), \quad R_a f(h) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(h \cdot e^{tT_a}) \qquad \forall f \in C^\infty(\mathrm{ISO}(2,1)), \tag{4}$$

where $e : \mathfrak{iso}(2,1) \to \mathrm{ISO}(2,1)$, $\boldsymbol{x} \mapsto e^{\boldsymbol{x}}$ is the exponential map for $\mathrm{ISO}(2,1)$. For the basis $\{J_a, P_a\}_{a=0,1,2}$, we denote by P^L_a, P^R_a, respectively, the right- and left-invariant vector fields associated with the translations and by J^L_a, J^R_a the ones associated with the Lorentz transformations. The former act trivially on functions that depend only on the Lorentzian component of $\mathrm{ISO}(2,1)$. For the latter, the action on such functions coincides with the action of the right- and left-invariant vector fields of the Lorentz group. The action of these vector fields on the coordinate functions $j^a : \mathrm{ISO}(2,1) \to \mathbb{R}$, $(u, -\mathrm{Ad}(u)\boldsymbol{q}) \mapsto q^a$ is given by

$$\left.\begin{aligned} &P^L_a j^b(u, -\mathrm{Ad}(u)\boldsymbol{j}) = \mathrm{Ad}(u)_a{}^b, && P^R_a j^b(u, -\mathrm{Ad}(u)\boldsymbol{j}) = -\delta_a{}^b, \\ &J^L_a j^b(u, -\mathrm{Ad}(u)\boldsymbol{j}) = 0, && J^R_a j^b(u, -\mathrm{Ad}(u)\boldsymbol{j}) = -\varepsilon^{ab}{}_c j^c. \end{aligned}\right\} \tag{5}$$

3 Physics background and motivation

3.1 Chern-Simons theory with gauge group ISO(2, 1) and the moduli space of flat ISO(2, 1)-connections

In the following, we consider Chern-Simons theory with gauge group $\mathrm{ISO}(2,1)$ on manifolds of topology $M \approx \mathbb{R} \times S_{g,n}$, where $S_{g,n}$ is an oriented surface of genus g with n punctures. In the absence of punctures, the solutions of the theory are flat connections A on an $\mathrm{ISO}(2,1)$-principal bundle over M. The punctures of $S_{g,n}$ are incorporated [14] into the theory by assigning the coadjoint orbit of an element $\mathcal{D}_i \in \mathfrak{iso}(2,1)$ to the i-th puncture and coupling it minimally to the connection A. Parametrising the coadjoint orbit of $\mathcal{D}_i$ in terms of group-valued functions $h_i : \mathbb{R} \to \mathrm{ISO}(2,1)$, one obtains the following expression for the Chern-Simons action:

$$S(A) = \int_M \langle A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A \rangle - 2 \sum_{i=1}^{n} \int_{\mathbb{R}} \langle \mathcal{D}_i, h_i^{-1} A|_{l_i} h_i + h_i^{-1} \mathrm{d}h_i \rangle \mathrm{d}t, \qquad (6)$$

where $\langle\;\rangle$ denotes the non-degenerate Ad-invariant symmetric bilinear form (2) on $\mathfrak{iso}(2,1)$ and $l_i : \mathbb{R} \to M$, $i = 1, \ldots, n$, are the curves defined by the punctures. Up to a topological term, the Chern-Simons action is invariant under gauge transformations $\gamma \in C^\infty(M, \mathrm{ISO}(2,1))$ that are constant along l_i: $A \mapsto \gamma^{-1} A \gamma + \gamma^{-1} \mathrm{d}\gamma$, $h_i \mapsto \gamma(l_i)^{-1} h_i$.

The connections that extremise the action (6) are those that are flat everywhere on M except at l_i ($i = 1, \ldots, n$), where their curvature $F \equiv \mathrm{d}A + A \wedge A$ develops δ-singularities. From the Hamiltonian formulation of the theory one then obtains that the gauge-invariant phase space of the theory is the moduli space $\mathcal{P}$ of flat $\mathrm{ISO}(2,1)$-connections on $S_{g,n}$ modulo gauge transformations [36, 14].

A convenient parametrisation of the moduli space is given by group homomorphisms $h : \pi_1(S_{g,n}) \to \mathrm{ISO}(2,1)$ that map the homotopy equivalence class m_i of a loop around the i-th puncture to the associated conjugacy class

$$\mathcal{C}_i = \{h \cdot \exp(\mathcal{D}_i) \cdot h^{-1} \mid h \in \mathrm{ISO}(2,1)\}. \qquad (7)$$

Two such group homomorphisms describe gauge-equivalent connections if and only if they are related by conjugation with an element of $\mathrm{ISO}(2,1)$. This implies that the moduli space of flat $\mathrm{ISO}(2,1)$-connections on $S_{g,n}$ is given by

$$\begin{aligned}\mathcal{P} &= \mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_n}\left(\pi_1(S_{g,n}), \mathrm{ISO}(2,1)\right)/\,\mathrm{ISO}(2,1) \qquad (8)\\ &= \{h \in \mathrm{Hom}(\pi_1(S_{g,n}), \mathrm{ISO}(2,1)) \mid h(m_i) \in \mathcal{C}_i\}/\,\mathrm{ISO}(2,1).\end{aligned}$$

The fundamental group $\pi_1(S_{g,n})$ of an oriented genus g surface $S_{g,n}$ with n punctures is generated by the homotopy equivalence classes of a loop m_i ($i = 1, \ldots, n$) around each puncture and the a- and b-cycles a_j, b_j ($j = 1, \ldots, g$) of each handle as shown in Figure 1. It has a single defining relation, which states that the curve c in Figure 1 is contractible:

$$\pi_1(S_{g,n}) = \langle m_1, \ldots, m_n, a_1, b_1, \ldots, a_g, b_g \mid b_g a_g^{-1} b_g^{-1} a_g \cdots b_1 a_1^{-1} b_1^{-1} a_1 m_n \cdots m_1 = 1 \rangle.$$

By characterising the group homomorphisms in (8) in terms of the images of the generators of $\pi_1(S_{g,n})$, we can thus identify the moduli space of flat connections with the set

$$\begin{aligned}\mathcal{P} = \{&(M_1, \ldots, M_n, A_1, B_1, \ldots, A_g, B_g) \in \mathrm{ISO}(2,1)^{n+2g} \mid \\ &M_i \in \mathcal{C}_i,\; [B_g, A_g^{-1}] \cdot [B_1, A_1^{-1}] \cdot M_n \cdots M_1 = 1\}/\,\mathrm{ISO}(2,1), \qquad (9)\end{aligned}$$

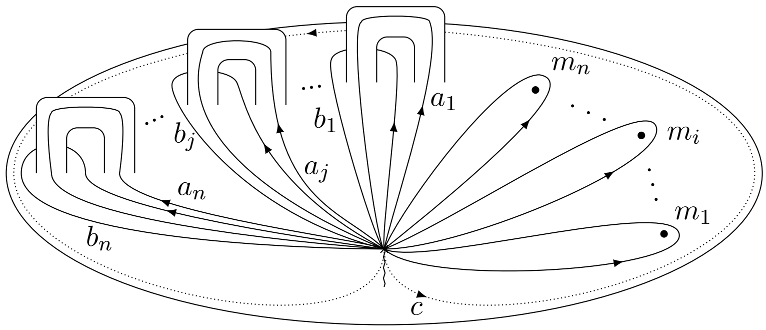

Figure 1: Generators of the fundamental group for an n-punctured genus g surface $S_{g,n}$. The chosen generators of the fundamental group $\pi_1(S_{g,n})$ are the homotopy equivalence classes of the curves $m_1, \ldots, m_n, a_1, b_1, \ldots, a_n, b_n$. The short wavy line indicates the cilium that defines a linear ordering of the incident edges at the basepoint.

where $[B_g, A_g^{-1}] = B_g \cdot A_g^{-1} \cdot B_g^{-1} \cdot A_g$ denotes the group commutator and the quotient is taken with respect to the diagonal action of $\mathrm{ISO}(2,1)$ on $\mathrm{ISO}(2,1)^{n+2g}$. In the gauge-theoretical description, the group elements $M_1, \ldots, B_g \in \mathrm{ISO}(2,1)$ correspond to the path-ordered exponentials of the gauge field A along the closed curves $m_1, \ldots, b_g$ displayed in Figure 1. In the following, we will sometimes refer to these group elements as holonomies.

3.2 Symplectic structure of the moduli space of flat connections

The moduli space of flat $\mathrm{ISO}(2,1)$-connections carries a canonical symplectic structure [7] that is obtained via symplectic reduction from the canonical symplectic structure on the space of connections on $S_{g,n}$. A convenient and explicit description of this symplectic structure is given in the works of Alekseev and Malkin and Fock and Rosly [22, 5]. They describe the canonical symplectic structure on the moduli space $\mathcal{P}$ in terms of a (non-canonical) Poisson structure on an enlarged ambient space $\mathcal{P}_{\mathrm{ext}}$. Via symplectic reduction, this Poisson structure then induces the canonical symplectic structure on $\mathcal{P}$.

In the following, we will work with a specific form of the Poisson structure in [22] which is associated with a choice of an ordered set of generators of the fundamental group $\pi_1(S_{g,n})$. It plays an important role as a starting point for the combinatorial quantisation of the theory [3, 4, 6, 12]. In this description, the ambient space $\mathcal{P}_{\mathrm{ext}}$ is given by $n + 2g$ copies of the Poincaré group, $\mathcal{P}_{\mathrm{ext}} = \mathrm{ISO}(2,1)^{n+2g}$, each corresponding to the holonomy along a generator of the fundamental group $\pi_1(S_{g,n})$.

Explicit expressions for this Poisson structure are given in Definition 4.1 in the next section. Here, we only discuss its most important structural features. Firstly, the definition of the bracket requires a classical r-matrix for the Lie algebra $\mathfrak{iso}(2,1)$, i.e. an element $r \in \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ that is a solution of the classical Yang-Baxter equation

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,$$

where $r_{12} = r^{\alpha\beta} T_\alpha \otimes T_\beta \otimes 1$, $r_{13} = r^{\alpha\beta} T_\alpha \otimes 1 \otimes T_\beta$, $r_{23} = r^{\alpha\beta} 1 \otimes T_\alpha \otimes T_\beta$. This property is needed to ensure that the bracket satisfies the Jacobi identity. Moreover, it is shown in [22] that this Poisson structure induces a symplectic structure on the moduli space $\mathcal{P}$, which

agrees with its canonical symplectic structure if and only if the symmetric part of r is dual to the pairing (2) in the Chern-Simons action:

$$r_S \equiv r_{(s)}^{\alpha\beta} T_\alpha \otimes T_\beta = \tfrac{1}{2}(P_a \otimes J^a + J^a \otimes P_a). \tag{10}$$

Note that Fock and Rosly's Poisson structure on the ambient space $\mathcal{P}_{\text{ext}}$ is therefore non-canonical in two ways. Firstly, it depends on the choice of a set of generators of the fundamental group $\pi_1(S_{g,n})$ and of a linear ordering of the incident edges at the basepoint. This ordering of the edges is indicated in Figure 1 by the short wavy line (cilium) and gives rise to a partial ordering of the generators of $\pi_1(S_{g,n})$: $m_1 < \cdots < m_n < a_1, b_1 < \cdots < a_g, b_g$.

Secondly, the definition of the Poisson structure on the ambient space $\mathcal{P}_{\text{ext}}$ requires the choice of a classical r-matrix, which is generally not unique; for a classification of the classical r-matrices for $\mathfrak{iso}(2,1)$ see [35]. However, it is shown in [22] that the Poisson structures on $\mathrm{ISO}(2,1)^{n+2g}$ associated with different choices of generators and orderings of $\pi_1(S_{g,n})$ and different choices of classical r-matrices that satisfy (10) induce the same symplectic structure on the moduli space $\mathcal{P}$. This is apparent in formula (12), which shows that the Poisson bracket of two functions on $\mathrm{ISO}(2,1)^{n+2g}$ depends only on the symmetric component of r if one of the two functions is invariant under the diagonal action of $\mathrm{ISO}(2,1)$ on $\mathrm{ISO}(2,1)^{n+2g}$.

In the following, we work with the classical r-matrix that corresponds to the structure of $\mathfrak{iso}(2,1)$ as a classical double of $\mathfrak{so}(2,1)$. In terms of the basis (1) it is given by $r = P_a \otimes J^a$. It is shown in [30], see also the discussion in Section 4.1, that Fock and Rosly's Poisson structure for this case can be formulated in terms of functions and certain vector fields on $\mathrm{SO}_+(2,1)^{n+2g}$ such that the Poisson bracket takes the form

$$\{f, g\} = 0, \quad \{X, f\} = \mathcal{L}_X f, \quad \{X, Y\} = [X, Y],$$

where $f, g \in C^\infty(\mathrm{SO}_+(2,1)^{n+2g})$, $X, Y \in \mathrm{Vec}(\mathrm{SO}_+(2,1)^{n+2g})$, $\mathcal{L}_X f$ denotes the Lie derivative and $[X, Y]$ the Lie bracket of vector fields on $\mathrm{SO}_+(2,1)^{n+2g}$.

This implies that the associated Poisson algebra has a canonical $\mathbb{N}$-grading, in which the subspaces of homogeneous degree are given by homogeneous polynomials in the vector fields X with $C^\infty(\mathrm{SO}_+(2,1)^{n+2g})$-valued coefficients. This $\mathbb{N}$-grading corresponds naturally to a physical dimension of $\hbar$ and plays an important role in the quantisation of the theory.

3.3 Quantisation

The description of the moduli space of flat connections outlined in the previous section serves as the starting point of the combinatorial quantisation formalism [3, 4, 6, 12] for Chern-Simons theories with compact, semisimple gauge groups. This formalism proceeds by quantising the auxiliary Poisson structure on the ambient space $\mathcal{P}_{\text{ext}}$ and then imposing the constraints in the quantum theory. As this auxiliary Poisson structure is closely related to certain Poisson structures from the theory of Poisson-Lie groups, the corresponding quantum algebra is given in terms of quantum groups. The implementation of the constraints in the quantum theory then reduces to a problem from the representation theory of the associated quantum group, namely to determining the invariant subspace in the tensor product of certain representations of this quantum group.

While this formalism is well-established for Chern-Simons theories with compact, semisimple gauge groups, for which the corresponding quantum groups are universal enveloping algebras

at a root of unity, it cannot be extended straightforwardly to Chern-Simons theories with non-compact gauge groups. This is due to the fact that the representations of the corresponding quantum groups no longer form a semisimple ribbon category and their characters become distributions.

The generalisation of the combinatorial quantisation formalism to Chern-Simons theory with gauge group $\mathrm{SL}(2,\mathbb{C})$ has been achieved in [11], for partial results on semidirect product gauge groups see [30, 27, 26]. However, there is currently no general quantisation formalism for moduli spaces of flat connections associated with non-compact or non-semisimple groups. Other quantisation approaches such as Reshetikhin-Turaev invariants [33] face similar problems. Although there is work that investigates their extension by analytic continuation [8, 40], there is no general method that allows one to extend these models to general non-compact or non-semisimple gauge groups.

For this reason, we pursue a different strategy, namely we implement the constraints directly in the classical theory by means of the Dirac gauge fixing procedure. This potentially avoids the issues with the implementation of the constraints in the quantisation of the theory and leads to an explicit description of the canonical Poisson structure on the moduli space of flat connections in terms of holonomies. In the following, we apply this approach to the description of the moduli space of flat $\mathrm{ISO}(2,1)$-connections in terms of the auxiliary Poisson structure on the ambient space $\mathcal{P}_{\text{ext}}$.

A specific example of such a gauge fixing procedure is investigated in [28]. It is shown there that in the application to (2+1)-gravity, this gauge fixing procedure has a direct physical interpretation. The gauge fixing conditions can be viewed as a prescription that specifies an observer with respect to the geometry of the spacetime. The resulting gauge-fixed Poisson structure then depends on two variables that correspond to the total mass and internal angular momentum of the spacetime as measured by this observer.

3.4 The Dirac gauge fixing procedure

In general, the Dirac gauge fixing procedure applies to constrained dynamical systems, i.e. to Poisson manifolds $(\mathcal{P}_{\text{ext}}, \{,\})$ with a set of constraint functions $\{\phi_i\}_{i=1,\dots,k} \subset C^\infty(\mathcal{P}_{\text{ext}})$. In the following, we restrict attention to the case where the constraints are first-class and such that 0 is a regular value of $\Phi = (\phi_1, \dots \phi_k) : \mathcal{P}_{\text{ext}} \to \mathbb{R}^k$. Gauge fixing then amounts to imposing an additional set of constraints $\{\chi_j\}_{j=1,\dots,k} \subset C^\infty(\mathcal{P}_{\text{ext}})$, the gauge fixing conditions, which must satisfy the following requirements [24]:

1. It is possible to map any point $q \in \{p \in \mathcal{P}_{\text{ext}} \mid \phi_i(p) = 0 \ \forall i = 1, \dots, k\}$ to a point on the constraint surface $\Sigma \coloneqq \{p \in \mathcal{P}_{\text{ext}} \mid \phi_i(p) = 0 \text{ and } \chi_i(p) = 0 \ \forall i = 1, \dots, k\}$ with the flows that the first-class constraints ϕ_i generate via the Poisson bracket.

2. The matrix $C = (\{\chi_j, \phi_i\})_{i,j=1,\dots,k}$ is invertible everywhere on the constraint surface Σ.

The second condition implies that the gauge fixing conditions $\{\chi_j\}_{j=1,\dots,k}$ together with the original constraints $\{\phi_i\}_{i=1,\dots,k}$ can be collected in a single set $\{C_i\}_{i=1,\dots,2k}$ of constraints such that 0 is a regular value of the function $C = (C_1, \dots, C_{2k}) : M \to \mathbb{R}^{2k}$ and for which the Dirac matrix $D = (D_{ij})_{i,j=1,\dots,2k}$, $D_{ij} \coloneqq \{C_i, C_j\}$, is invertible anywhere on the constraint surface

Σ. The Dirac bracket of two functions $f, g \in C^\infty(\Sigma)$ is then defined by

$$\{f, g\}_D \coloneqq \{\tilde{f}, \tilde{g}\} - \sum_{i,j=1}^{2k} \{\tilde{f}, C_i\}(D^{-1})_{ij}\{C_j, \tilde{g}\},$$

where $\tilde{f}, \tilde{g} \in C^\infty(\mathcal{P}_{\text{ext}})$ are arbitrary extensions of $f, g \in C^\infty(\Sigma)$. The Dirac bracket does not depend on the choice of the extensions $\tilde{f}, \tilde{g}$ and defines a Poisson structure on Σ [15, 16, 24].

The physical interpretation of this bracket can be summarised as follows: The Poisson manifold $(\mathcal{P}_{\text{ext}}, \{\,,\,\})$ plays the role of an extended, non-gauge-invariant phase space which contains redundant degrees of freedom corresponding to different descriptions of a single physical state. The gauge-invariant or physical phase space is given as the quotient $\mathcal{P} = \mathcal{Q}/\!\sim$, where $\mathcal{Q} \coloneqq \{p \in \mathcal{P}_{\text{ext}} \mid \phi_i(p) = 0 \; \forall i = 1, \ldots, k\}$ and two points on $\mathcal{Q}$ are identified if they are mapped into each other by the flows the first-class constraints ϕ_i generate via the the Poisson bracket. The associated equivalence classes are called gauge orbits.

Imposing gauge fixing conditions amounts to selecting a representative in each gauge orbit. The first requirement on the gauge fixing conditions ensures that the gauge fixing conditions select at least one representative in every gauge orbit. The second requirement ensures that they select at most one representative in each gauge orbit. Imposing gauge fixing conditions thus amounts to constructing a diffeomorphism that identifies the quotient $\mathcal{P} = \mathcal{Q}/\!\sim$ with the submanifold $\Sigma \subset \mathcal{Q} \subset \mathcal{P}_{\text{ext}}$.

From the viewpoint of symplectic reduction, this procedure can be interpreted in the following way. If $(\mathcal{P}_{\text{ext}}, \{\;\})$ is symplectic, then the submanifold $\mathcal{Q} \subset \mathcal{P}_{\text{ext}}$ is coisotropic, and the flows generated by the first-class constraints ϕ_i define a foliation of $\mathcal{Q}$. If we think of $\mathcal{Q}$ as a bundle over $\mathcal{P}$, then choosing a representative for each equivalence class in $\mathcal{P}$ amounts to specifying a global section on $\mathcal{Q}$. This is achieved via the gauge fixing conditions, which define a diffeomorphism $\xi : \Sigma \subset \mathcal{P}_{\text{ext}} \to \mathcal{P}$, $p \mapsto [p]$. The manifold $\mathcal{P}$ carries a canonical symplectic structure obtained via symplectic reduction from the symplectic structure on $\mathcal{P}_{\text{ext}}$. The pull-back of this symplectic structure to the submanifold $\Sigma \subset \mathcal{P}_{\text{ext}}$ with ξ is the Dirac bracket on Σ.

The situation is similar in the case where $(\mathcal{P}_{\text{ext}}, \{\;\})$ is not symplectic but there is a function $\Psi = (\psi_1, \ldots, \psi_l) \in C^\infty(\mathcal{P}_{\text{ext}}, \mathbb{R}^l)$ such that 0 is a regular value of Ψ, $\Psi^{-1}(0)$ a symplectic submanifold of $\mathcal{P}_{\text{ext}}$ and $\psi_1, \ldots, \psi_l$ Poisson-commute with all functions on $\mathcal{P}_{\text{ext}}$. In this case, the reasoning above can be applied to the submanifold $\Psi^{-1}(0)$.

3.5 Constraints and gauge fixing conditions for the moduli space

The moduli space of flat $\mathrm{ISO}(2,1)$-connections can be viewed as a constrained system in the sense of Dirac. From this viewpoint, the Poisson manifold $(\mathcal{P}_{\text{ext}}, \{\,,\})$ is identified with the ambient space $\mathcal{P}_{\text{ext}} = \mathrm{ISO}(2,1)^{n+2g}$ equipped with Fock and Rosly's Poisson structure [22]. From expression (9) for the moduli space of flat $\mathrm{ISO}(2,1)$-connections, it is then apparent that the moduli space is obtained from $\mathcal{P}_{\text{ext}}$ by imposing a group-valued constraint that arises from the defining relation of the fundamental group $\pi_1(S_{g,n})$, together with a set of constraints that restrict the holonomies $M_1, \ldots, M_n$ to the conjugacy classes (7). The latter can be formulated as pairs of constraints of the form

$$\mathrm{Tr}(u_{M_i}) - c_i \approx 0, \qquad \mathrm{Tr}(j^a_{M_i} J_a \cdot u_{M_i}) - d_i \approx 0,$$

with real parameters c_i, d_i for each puncture. It turns out that these constraints are Casimir functions of the Poisson bracket, i.e. Poisson-commute with all functions on $\mathcal{P}_{\text{ext}}$. For this reason, reducing the Poisson structure to the relevant conjugacy classes presents no difficulties and does not require any gauge fixing.

The group-valued constraint from the defining relation of the fundamental group $\pi_1(S_{g,n})$ can be viewed as a set of six first-class constraints for the Fock-Rosly bracket [28]. Parametrising the holonomy of the curve c in Figure 1 as

$$(u_C^{-1}, \boldsymbol{j}_C) \coloneqq M_1^{-1} \cdots M_n^{-1} [A_1^{-1}, B_1] \cdots [A_g^{-1}, B_g],$$

one can express this group-valued constraint in the form of the six constraints

$$\mathrm{Tr}(J_a \cdot u_C) \approx 0, \qquad j_C^a \approx 0 \qquad \forall a \in \{0, 1, 2\}. \tag{11}$$

The Poisson brackets of these six constraint functions are closely related to the Lie bracket of $\mathfrak{iso}(2,1)$. The associated gauge transformations which they generate via the Poisson bracket are given by the diagonal action of $\mathrm{ISO}(2,1)$ on $\mathrm{ISO}(2,1)^{n+2g}$.

A specific choice of gauge fixing conditions for the constraints (11) is investigated in [28]. It imposes gauge fixing conditions on the holonomies associated with two punctures on the surface $S_{g,n}$ and derives the associated Dirac bracket. In this paper we consider general gauge fixing conditions, subject to certain structural requirements, and investigate the resulting Dirac brackets. We require that the gauge fixing conditions satisfy the conditions 1 and 2 in Section 3.4 and are subject to the following two additional restrictions:

a. The gauge fixing conditions depend only on the holonomies M_i, M_j associated with two punctures on the surface $S_{g,n}$.

b. The gauge fixing conditions depend at most linearly on the variables j_{M_i} and j_{M_j} associated with these holonomies.

The first condition is motivated by convenience and by physics considerations. Although it is feasible in principle to impose gauge fixing conditions that involve the holonomies of more than two punctures, this would complicate many details of the description without adding much on the conceptual level. Moreover, in the application to (2+1)-gravity, gauge fixing conditions based on the holonomies of two punctures have a direct physical interpretation, while the interpretation of a complicated gauge fixing condition involving more than two punctures is less obvious.

Note also that the first condition allows us to restrict attention to gauge fixing conditions that depend only on the holonomies M_1, M_2 of the first two punctures on $S_{g,n}$. This is due to the fact that different orderings of the punctures are related by the action of the braid group on $S_{g,n}$ and the braid group on the associated surface $S_{g,n} \setminus D$ with a disc removed [9]. It is shown in [31] that the braid group of the surface $S_{g,n} \setminus D$ acts by Poisson isomorphisms on the Poisson manifold $(\mathcal{P}_{\text{ext}} = \mathrm{ISO}(2,1)^{n+2g}, \{\,,\,\})$. The action of the braid group thus allows one to permute the punctures and to suppose that the gauge fixing conditions depend only on the holonomies of the first two punctures.

The second condition is motivated by structural considerations, namely the wish to preserve the natural $\mathbb{N}$-grading of the Poisson structure. As we will see in the following, gauge fixing conditions that are non-linear in the variables j_{M_1} or j_{M_2} and, consequently, non-linear in

the vector fields on $\mathrm{SO}_+(2,1)^{n+2g}$, would compromise this grading. However, the grading is an important structural feature of the theory and plays a central role in its quantisation [30, 27, 26]. For this reason, it seems natural to impose that it is preserved by the gauge fixing procedure.

Conditions 1 and 2 from Section 3.4 together with the additional assumptions a and b above imply that the gauge fixing conditions can be brought into the form

$$\sum_{i=1}^{2} \Theta_a^{M_i,1}\, j^a_{M_i} \approx 0, \quad \sum_{i=1}^{2} \Theta_a^{M_i,2}\, j^a_{M_i} \approx 0, \quad \sum_{i=1}^{2} \Theta_a^{M_i,3}\, j^a_{M_i} \approx 0, \quad \Delta_1 \approx 0, \quad \Delta_2 \approx 0, \quad \Delta_3 \approx 0,$$

where $\Theta_a^{M_i,j}, \Delta_j \in C^\infty(\mathrm{SO}_+(2,1) \times \mathrm{SO}_+(2,1))$ and the two copies of the Lorentz group $\mathrm{SO}_+(2,1)$ are identified with the Lorentzian components of the holonomies M_1 and M_2. These gauge fixing conditions allow one to express the two constrained holonomies M_1 and M_2 as functions of the four fixed parameters that characterise the conjugacy classes $\mathcal{C}_1, \mathcal{C}_2$ and of two conjugation-invariant dynamical variables ψ, α, which depend only on the product $M_2 \cdot M_1$. As there are many possible definitions of these variables, we will not adhere to one of them, but impose that they are given in terms of the Lorentzian and translational components of the product $M_2 \cdot M_1 = (u_{12}, -\mathrm{Ad}(u_{12})\boldsymbol{j}_{12})$ as

$$\psi = f(\mathrm{Tr}(u_{12})), \quad \alpha = g(\mathrm{Tr}(u_{12}))\,\mathrm{Tr}(j^a_{12} J_a \cdot u_{12}) + h(\mathrm{Tr}(u_{12})),$$

with diffeomorphisms $f, g \in C^\infty(\mathbb{R})$ and a smooth function $h \in C^\infty(\mathbb{R})$.

As for the gauge fixing conditions investigated in [28], the Dirac gauge fixing procedure with these gauge fixing conditions gives rise to a Poisson structure $\{\,,\}_D$ on the constraint surface $\Sigma \subset \mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$, where $\mathbb{R}^2$ is parametrised by the variables ψ, α and $\mathrm{ISO}(2,1)^{n-2+2g}$ by the non-gauge-fixed holonomies $M_3, \ldots, B_g$. This Poisson structure is derived in the next section.

4 Gauge fixing and solutions of the classical dynamical Yang-Baxter equation

4.1 General form of the Dirac bracket

In this section, we derive explicitly the Dirac bracket obtained by gauge fixing the auxiliary Poisson structure from [22] on the ambient space $\mathrm{ISO}(2,1)^{n+2g}$. While the Poisson structure in [22] is associated with general ciliated fat graphs on a genus g-surface $S_{g,n}$ with n punctures, we restrict attention to the case where the graph is a set of generators of the fundamental group $\pi_1(S_{g,n})$ as depicted in Figure 1. In that case, the Poisson structure from [22] takes the following form.

Definition 4.1 ([22])**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$, $\{T_\alpha\}_{\alpha=1,\ldots,dim(G)}$ a basis of $\mathfrak{g}$ and $r = r^{\alpha\beta} T_\alpha \otimes T_\beta \in \mathfrak{g} \otimes \mathfrak{g}$. Then Fock and Rosly's bivector $B_r^{n,g}$ is the antisymmetric section of the bundle $TG^{n+2g} \otimes TG^{n+2g}$ defined by*

$$B_r^{n,g} = \tfrac{1}{2} r_{(a)}^{\alpha\beta} \Big(\sum_{i=1}^{n+2g} L^i_\alpha + R^i_\alpha\Big) \otimes \Big(\sum_{j=1}^{n+2g} L^j_\beta + R^j_\beta\Big) + \tfrac{1}{2} r_{(s)}^{\alpha\beta} \sum_{1\leq i<j\leq n+2g} \Big(L^i_\alpha + R^i_\alpha\Big) \wedge \Big(L^j_\beta + R^j_\beta\Big) \tag{12}$$
$$+ \tfrac{1}{2} r_{(s)}^{\alpha\beta} \Big(\sum_{i=1}^{n} R^i_\alpha \wedge R^i_\beta + \sum_{j=1}^{g} \Big[L^{n+2j}_\alpha \wedge L^{n+2j}_\beta - (R^{n+2j-1}_\alpha + L^{n+2j-1}_\alpha) \wedge R^{n+2j}_\beta - L^{n+2j-1}_\alpha \wedge L^{n+2j}_\beta \Big] \Big),$$

where L^i_α *and* R^i_α *denote the right- and left-invariant vector fields associated with the different components of* G^{n+2g} *and the basis elements* T_α*:*

$$L^i_\alpha f(g_1,\ldots,g_{n+2g}) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(u_1,\ldots,u_{i-1},e^{-tT_\alpha}\cdot u_i,u_{i+1},\ldots,u_{n+2g}),$$
$$R^i_\alpha f(g_1,\ldots,g_{n+2g}) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(u_1,\ldots,u_{i-1},u_i\cdot e^{tT_\alpha},u_{i+1},\ldots,u_{n+2g}),$$

and $r^{\alpha\beta}_{(a)}$, $r^{\alpha\beta}_{(s)}$ *the coefficients of the antisymmetric and symmetric component of* r*:*

$$r^{\alpha\beta}_{(a)} = \tfrac{1}{2}(r^{\alpha\beta} - r^{\beta\alpha}) \qquad r^{\alpha\beta}_{(s)} = \tfrac{1}{2}(r^{\alpha\beta} + r^{\beta\alpha}).$$

The associated bracket on G^{n+2g} *is the antisymmetric bilinear map* $\{\ \} : C^\infty(G^{n+2g}) \times C^\infty(G^{n+2g}) \to C^\infty(G^{n+2g})$ *given by*

$$\{f,g\} = B^{n,g}_r(\mathrm{d}f \otimes \mathrm{d}g) \qquad \forall f,g \in C^\infty(G^{n+2g}).$$

The main advantage of this description is that it defines an auxiliary Poisson structure on G^{n+2g} that is given in terms of a classical r-matrix for $\mathfrak{g}$ and induces the canonical symplectic structure on the moduli space of flat G-connections on $S_{g,n}$.

Theorem 4.2 ([22])**.** *If* r *is a solution of the classical Yang-Baxter equation*

$$[[r,r]] = [r_{12},r_{13}] + [r_{12},r_{23}] + [r_{13},r_{23}] = 0,$$

then $B^{n,g}_r$ *defines a Poisson structure on* G^{n+2g}*. If additionally* $\langle\ ,\ \rangle$ *is a non-degenerate* Ad*-invariant symmetric bilinear form on* $\mathfrak{g}$ *and*

$$r^{\alpha\beta}_{(s)} = \tfrac{1}{2}\kappa^{\alpha\beta} \qquad \text{with } \kappa^{\alpha\beta}\kappa_{\beta\gamma} = \delta^\alpha{}_\gamma,\ \kappa_{\alpha\beta} = \langle T_\alpha, T_\beta\rangle,$$

then the Poisson structure defined by $B^{n,g}_r$ *induces the canonical symplectic structure on the moduli space of flat* G*-connections on* $S_{g,n}$*.*

It is shown in [5] that this Poisson structure can be identified with the direct product of n copies of the dual Poisson-Lie structure on G and g copies of the Heisenberg double Poisson structure associated with G and r.

For the case at hand, where the Lie group is the Poincaré group in three dimensions, $G = \mathrm{ISO}(2,1)$, a classical r-matrix is given by $r = P_a \otimes J^a$. The corresponding Poisson structure on $\mathrm{ISO}(2,1)^{n+2g}$ is computed explicitly in [29], and in [30] it is shown that this Poisson structure can be identified with the direct product of n copies of the dual Poisson-Lie structure on $\mathrm{ISO}(2,1)$ and $2g$ copies of the cotangent bundle Poisson structure:

$$(\mathrm{ISO}(2,1)^{n+2g},\{\}) = \underbrace{\mathrm{ISO}(2,1)^* \times \ldots \times \mathrm{ISO}(2,1)^*}_{n\times} \times \underbrace{T^*(\mathrm{SO}_+(2,1)) \times \ldots \times T^*(\mathrm{SO}_+(2,1))}_{2g\times}.$$

From this description, it is directly apparent that the symplectic leaves of this Poisson structure are of the form $\mathcal{C}_1 \times \cdots \times \mathcal{C}_n \times \mathrm{ISO}(2,1)^{2g}$, where $\mathcal{C}_i \subset \mathrm{ISO}(2,1)$ are fixed conjugacy classes.

In the following, we will not use this identification, but we will work with a description of the Poisson structure that is closer to the formula in Definition 4.1. This formulation has the advantage that it is more adapted to physics applications, especially in the Chern-Simons

formulation of $(2+1)$-gravity, and that its geometrical interpretation is more apparent. To emphasise the geometrical interpretation of the variables and their relation with the set of generators of the fundamental group $\pi_1(S_{g,n})$ in Figure 1 we denote elements of $\mathrm{ISO}(2,1)^{n+2g}$ and $\mathrm{SO}_+(2,1)^{n+2g}$ as, respectively,

$$(M_1,\ldots,M_n,A_1,B_1,\ldots,A_g,B_g)\in \mathrm{ISO}(2,1)^{n+2g}, \qquad (u_{M_1},\ldots,u_{B_g})\in \mathrm{SO}_+(2,1)^{n+2g},$$

and write $J^{L,a}_X$, $J^{R,a}_Y$ for the associated right- and left-invariant vector fields on $\mathrm{SO}_+(2,1)^{n+2g}$:

$$\begin{aligned}
J^{L,a}_X f(u_{M_1},\ldots,u_{B_g}) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(u_{M_1},\ldots,e^{-tJ_a}\cdot u_X,\ldots,u_{B_g}),\\
J^{R,a}_X f(u_{M_1},\ldots,u_{B_g}) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(u_{M_1},\ldots,u_X\cdot e^{tJ_a},\ldots,u_{B_g}).
\end{aligned}$$

It is shown in [30] that by using the identification $\mathrm{ISO}(2,1) = T\,\mathrm{SO}_+(2,1)$ and the classical r-matrix $r = P_a\otimes J^a$, the Poisson structure given by (12) can be expressed in terms of functions $f\in C^\infty(\mathrm{SO}_+(2,1)^{n+2g})$ and certain vector fields on $\mathrm{SO}_+(2,1)^{n+2g}$. For this, one identifies the coordinate functions $j^a_X : (M_1,\ldots,B_g)\mapsto q^a_X$, where we use the parametrisation $X = (u_X, -\mathrm{Ad}(u_X)\boldsymbol{q})$ for $X\in\{M_1,\ldots,B_g\}$, with certain vector fields on $\mathrm{SO}_+(2,1)^{n+2g}$. The Poisson bracket of two variables j^a_X, j^b_Y then coincides with the Lie bracket of the associated vector fields, the Poisson bracket of a variable j^a_X with a function on $\mathrm{SO}_+(2,1)^{n+2g}$ coincides with its Lie derivative and any two functions on $\mathrm{SO}_+(2,1)^{n+2g}$ Poisson-commute.

Theorem 4.3 ([29, 30])**.** *For $G=\mathrm{ISO}(2,1)$ and $r=P_a\otimes J^a$, the Poisson structure (12) on $\mathrm{ISO}(2,1)^{n+2g} = T\,\mathrm{SO}_+(2,1)^{n+2g}$ is characterised uniquely in terms of vector fields j^a_X, $a\in\{0,1,2\}$, $X\in\{M_1,\ldots,B_g\}$ on $\mathrm{SO}_+(2,1)^{n+2g}$ and functions on $\mathrm{SO}_+(2,1)^{n+2g}$:*

$$\{f,g\}=0, \qquad \{j^a_X,f\}=\mathcal{L}_{j^a_X}f, \qquad \{j^a_X,j^b_Y\}=[j^a_X,j^b_Y] \qquad \forall f\in C^\infty(\mathrm{SO}_+(2,1)^{n+2g}),$$

where $\mathcal{L}$ denotes the Lie derivative and $[\ ,\ ]$ the Lie bracket on $\mathrm{SO}_+(2,1)^{n+2g}$. The vector fields j^a_X on $\mathrm{SO}_+(2,1)^{n+2g}$ are given by:

$$\begin{aligned}
j^a_{M_i} &= -\left(J^{R,a}_{M_i}+J^{L,a}_{M_i}\right) - (\mathbb{1}-\mathrm{Ad}(u_{M_i}))^a{}_b \sum_{Y>M_i}\left(J^{R,b}_Y + J^{L,b}_Y\right),\\
j^a_{A_j} &= -\left(J^{R,a}_{A_j}+J^{L,a}_{A_j}+J^{L,a}_{B_j}+(\mathbb{1}-\mathrm{Ad}(u_{A_j}^{-1}u_{B_j}))^a{}_b J^{R,b}_{B_j}\right) - (\mathbb{1}-\mathrm{Ad}(u_{A_j}))^a{}_b \sum_{Y>A_j}\left(J^{R,b}_Y + J^{L,b}_Y\right),\\
j^a_{B_j} &= -\left(J^{R,a}_{B_j}+J^{L,a}_{B_j}+J^{L,b}_{A_j}\right) - (\mathbb{1}-\mathrm{Ad}(u_{B_j}))^a{}_b \sum_{Y>A_j}\left(J^{R,b}_Y + J^{L,b}_Y\right),
\end{aligned}$$

where $Y>X$ refers to the partial ordering of the generators of $\pi_1(S_{g,n})$: $Y>X$ if $X=M_i$ and $Y=M_j$ with $i<j$ or if $X\in\{A_i,B_i\}$, $Y\in\{A_j,B_j\}$ with $i<j$ or if $X\in\{M_1,\ldots,M_n\}$, $Y\in\{A_1,B_1,\ldots,A_g,B_g\}$.

We will now determine the Dirac bracket associated with the Poisson structure on $\mathrm{ISO}(2,1)^{n+2g}$ and certain smooth constraint functions on $\mathrm{ISO}(2,1)^{n+2g}$. The Dirac bracket is a well-established formalism from the theory of constrained Hamiltonian systems and, in a certain sense, can be viewed as the Poisson counterpart or the physicist's version of symplectic reduction. A more detailed discussion of this is given in Section 3.4, for the general theory we refer the reader to [15, 16, 24].

Definition 4.4 ([15, 16])**.** *Let $(M, \{\,,\,\})$ be an n-dimensional Poisson manifold, $k < n$, and $C = (C_1, \ldots, C_k) : M \to \mathbb{R}^k$ a smooth function such that 0 is a regular value of C and the matrix $D(p) = (\{C_i, C_j\}(p))_{i,j=1,\ldots,k}$ is invertible for all $p \in \Sigma = C^{-1}(0)$. The* ***Dirac bracket*** *for C is the antisymmetric bilinear map $\{\,,\,\}_D : C^\infty(\Sigma) \times C^\infty(\Sigma) \to C^\infty(\Sigma)$ defined by*

$$\{f, g\}_D = \{\tilde{f}, \tilde{g}\}|_\Sigma - \sum_{i,j=1}^{k} \{\tilde{f}, C_i\} \cdot (D|_\Sigma)^{-1}_{ij} \cdot \{C_j, \tilde{g}\}|_\Sigma,$$

where $\tilde{f}, \tilde{g} \in C^\infty(M)$ are arbitrary extensions of $f, g \in C^\infty(\Sigma)$: $\tilde{f}|_\Sigma = f$, $\tilde{g}|_\Sigma = g$. The Dirac bracket does not depend on the choice of these extensions and defines a Poisson structure on Σ. The submanifold $\Sigma = C^{-1}(0) \subset M$ is called ***constraint surface****, the functions $C_i : M \to \mathbb{R}$ are called* ***constraint functions****.*

The aim is now to determine the Dirac bracket for the Poisson structure from Definition 4.1 for constraint functions that relate this Poisson structure to the moduli space of flat $\mathrm{ISO}(2,1)$-connections:

$$\begin{aligned} \mathcal{P} &= \{h \in \mathrm{Hom}(\pi_1(S_{g,n}), \mathrm{ISO}(2,1)) \mid h(m_i) \in \mathcal{C}_i\}/\,\mathrm{ISO}(2,1) \\ &\cong \{(M_1, \ldots, B_g) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \times \mathrm{ISO}(2,1)^{2g} \mid \\ &\qquad\qquad [B_g, A_g^{-1}] \cdots [B_1, A_1^{-1}] \cdot M_n \cdots M_1 = 1\}/\,\mathrm{ISO}(2,1). \end{aligned} \tag{13}$$

As discussed in Section 3.5, this leads to constraint functions of the form

$$\left.\begin{array}{lll} C_1 = j_C^0, & C_2 = j_C^1, & C_3 = j_C^2, \\ C_4 = \sum_{i=1}^2 \Theta_a^{M_i,1}\, j_{M_i}^a, & C_5 = \sum_{i=1}^2 \Theta_a^{M_i,2}\, j_{M_i}^a, & C_6 = \sum_{i=1}^2 \Theta_a^{M_i,3}\, j_{M_i}^a, \\ C_7 = \mathrm{Tr}(J_0 \cdot u_C), & C_8 = \mathrm{Tr}(J_1 \cdot u_C), & C_9 = \mathrm{Tr}(J_2 \cdot u_C), \\ C_{10} = \Delta_1, & C_{11} = \Delta_2, & C_{12} = \Delta_3, \end{array}\right\} \tag{14}$$

where $j_{M_i}^a$ is defined as in Theorem 4.3 and $\Theta_a^{M_i,j}, \Delta_j \in C^\infty(\mathrm{SO}_+(2,1) \times \mathrm{SO}_+(2,1))$ are functions that depend only on the $\mathrm{SO}_+(2,1)$-part of the first two copies of $\mathrm{ISO}(2,1)$ and

$$(u_C^{-1}, \boldsymbol{j}_C) \coloneqq M_1^{-1} \cdots M_n^{-1} \cdot [A_1^{-1}, B_1] \cdots [A_g^{-1}, B_g].$$

The functions $(C_i)_{i=1,2,3,7,8,9}$ have an interpretation as first-class constraints in the Dirac gauge fixing formalism and the functions $(C_i)_{i=4,5,6,10,11,12}$ play the role of gauge fixing conditions. While the former are fixed and implement the condition $[B_g, A_g^{-1}] \cdots [B_1, A_1^{-1}] \cdot M_n \cdots M_1 = 1$, the latter involve functions $\Theta_a^{M_i,j}, \Delta_j$ which can be chosen arbitrarily as long as the requirements from Definition 4.4 and the conditions a and b from Section 3.5 are met. The latter ensure that the constraint functions are adapted to the tangent bundle structure of $\mathrm{ISO}(2,1) = T\,\mathrm{SO}_+(2,1)$. Different choices of these functions correspond to different gauge choices. They implement the quotient by $\mathrm{ISO}(2,1)$ in (13) and restrict the variables M_1, M_2 in such a way that for all points $(M_1, \ldots, B_g) \in \Sigma = C^{-1}(0)$, the components $M_1, M_2 \in \mathrm{ISO}(2,1)$ are determined uniquely by two real parameters

$$\psi = f(\mathrm{Tr}(u_{12})), \quad \alpha = g(\mathrm{Tr}(u_{12}))\,\mathrm{Tr}(j_{12}^a J_a \cdot u_{12}) + h(\mathrm{Tr}(u_{12})), \tag{15}$$

where $f, g \in C^\infty(\mathbb{R})$ are arbitrary diffeomorphisms and $h \in C^\infty(\mathbb{R})$. This allows us to identify the constraint surface $\Sigma = C^{-1}(0)$ with a subset of $\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$, where the $\mathbb{R}^2$ is parametrised by (ψ, α) and $\mathrm{ISO}(2,1)^{n-2+2g}$ by $(M_3, \ldots, B_g)$.

Given the expressions for the Poisson structure on $\mathrm{ISO}(2,1)^{n+2g}$ and the constraint functions (14), we can explicitly compute the associated Dirac bracket and obtain a Poisson structure on $\Sigma \subset \mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$.

Theorem 4.5. *For all constraint functions of the form* (14) *that satisfy the requirements in Definition 4.4 and conditions a and b from Section 3.5, the associated Dirac bracket defines a Poisson structure* $\{\,,\,\}_D$ *on* $\Sigma \subset \mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$*, which takes the following form:*

1. *The Dirac bracket of* ψ *and* α *vanishes:* $\{\psi, \alpha\}_D = 0$.
2. *For all* $X \in \{M_3, \ldots, B_g\}$ *and* $f \in C^\infty(\mathrm{SO}_+(2,1)^{n-2+2g})$*:*

$$\begin{aligned} \{\psi, f\}_D &= 0, & \{\psi, \boldsymbol{j}_X\}_D &= -(\mathbb{1} - \mathrm{Ad}(u_X^{-1}))\, \boldsymbol{q}_\psi, \\ \{\alpha, f\}_D &= \sum_{Y \in \{M_3,\ldots,B_g\}} q_\alpha^a (J_a^{R,Y} + J_a^{L,Y}) f, & \{\alpha, \boldsymbol{j}_X\}_D &= -(\mathbb{1} - \mathrm{Ad}(u_X^{-1}))\boldsymbol{q}_\theta - \boldsymbol{q}_\alpha \wedge \boldsymbol{j}_X, \end{aligned}$$

   with $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta : \mathbb{R}^2 \to \mathbb{R}^3$ *satisfying* $\boldsymbol{q}_\psi \wedge \boldsymbol{q}_\alpha = 0$ *and* $\partial_\alpha \boldsymbol{q}_\psi = \partial_\alpha \boldsymbol{q}_\alpha = \partial_\alpha^2 \boldsymbol{q}_\theta = 0$.
3. *For* $F, G \in C^\infty(\mathrm{ISO}(2,1)^{n-2+2g})$*:*

$$\{F, G\}_D = B_r^{n-2,g}(\mathrm{d}F \otimes \mathrm{d}G),$$

   where $B_r^{n-2,g}$ *is the Poisson bivector* (12) *and* $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ *is given by*

$$r(\psi, \alpha) = P_a \otimes J^a - V^{bc}(\psi)(P_b \otimes J_c - J_c \otimes P_b) + \varepsilon^{bcd} m_d(\psi, \alpha) P_b \otimes P_c,$$

   where $V : \mathbb{R} \to \mathrm{Mat}(3, \mathbb{R})$ *and* $\boldsymbol{m} : \mathbb{R}^2 \to \mathbb{R}^3$ *satisfies* $\partial_\alpha^2 \boldsymbol{m} = 0$.

Proof. The proof is a direct generalisation of the proof of Theorem 5.1 in [28].

1. The Dirac matrix associated to the constraints (14) takes the form

$$D = \begin{pmatrix} J & P \\ -P^T & 0 \end{pmatrix} \quad \text{with} \quad J := (\{C_i, C_j\})_{i,j=1,\ldots,6},\ P := (\{C_i, C_{j+6}\})_{i,j=1,\ldots,6}.$$

   On the constraint surface, the (6×6)-matrices J and P can be expressed as

$$J|_\Sigma = \begin{pmatrix} 0 & H \\ -H^T & G \end{pmatrix}, \qquad P|_\Sigma = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix},$$

   with (3×3)-matrices A, B, C, G, H given by

$$\left.\begin{aligned} A_{ij} := \{C_i, C_{j+9}\}|_\Sigma, \quad B_{ij} := \{C_{i+3}, C_{j+6}\}|_\Sigma, \quad C_{ij} := \{C_{i+3}, C_{j+9}\}|_\Sigma, \\ G_{ij} := \{C_{i+3}, C_{j+3}\}|_\Sigma, \quad H_{ij} := \{C_i, C_{j+3}\}|_\Sigma, \end{aligned}\right\}\ i, j = 1, 2, 3.$$

   This implies that the inverse of the Dirac matrix D on the constraint surface is given by

$$(D|_\Sigma)^{-1} = \begin{pmatrix} 0 & -(P^{-1})^T \\ (P|_\Sigma)^{-1} & (P|_\Sigma)^{-1}(J|_\Sigma)(P|_\Sigma^{-1})^T \end{pmatrix} \text{ with } (P|_\Sigma)^{-1} = \begin{pmatrix} -B^{-1}CA^{-1} & B^{-1} \\ A^{-1} & 0 \end{pmatrix},$$

$$(P|_\Sigma)^{-1}(J|_\Sigma)(P|_\Sigma^{-1})^T = \begin{pmatrix} B^{-1}[G - CA^{-1}H + (CA^{-1}H)^T](B^{-1})^T & -B^{-1}H^T(A^{-1})^T \\ A^{-1}H(B^{-1})^T & 0 \end{pmatrix}.$$

Inserting these expression into the general formula in Definition 4.4, one finds that for all $X, Y \in \{M_1, \ldots, B_g\}$ and $f, g \in C^\infty(\mathrm{SO}_+(2,1)^{n+2g})$, the Dirac bracket takes the form

$$\{f, g\}_D = 0, \tag{16a}$$

$$\begin{aligned}\{j_X^a, f\}_D = \{j_X^a, f\}|_\Sigma + \sum_{i,j=1}^{3} \Big[&\{j_X^a, C_{i+6}\}(B^{-1})_{ij}\{f, C_{j+3}\} + \{j_X^a, C_{i+9}\}(A^{-1})_{ij}\{f, C_j\} \\ &- \{j_X^a, C_{i+6}\}(B^{-1}CA^{-1})_{ij}\{f, C_j\}\Big]|_\Sigma, \end{aligned} \tag{16b}$$

$$\begin{aligned}\{j_X^a, j_Y^b\}_D = \{j_X^a, j_Y^b\}|_\Sigma &+ \sum_{i=1}^{6}\sum_{j=7}^{12} \{j_X^a, C_i\}|_\Sigma (D|_\Sigma^{-1})_{ij}\{j_Y^b, C_j\}|_\Sigma \\ + \sum_{i=7}^{12}\sum_{j=1}^{6} \{j_X^a, C_i\}|_\Sigma (D|_\Sigma^{-1})_{ij}\{j_Y^b, C_j\}|_\Sigma &+ \sum_{i=7}^{12}\sum_{j=7}^{12} \{j_X^a, C_i\}|_\Sigma (D|_\Sigma^{-1})_{ij}\{j_Y^b, C_j\}|_\Sigma. \end{aligned} \tag{16c}$$

2. To prove the relations for the brackets involving ψ and α, we use (16) to compute the Dirac brackets of j_{M_1}, j_{M_2} and functions $g \in C^\infty(\mathrm{SO}_+(2,1) \times \mathrm{SO}_+(2,1))$ of the variables u_{M_1}, u_{M_2} with functions $f \in C^\infty(\mathrm{SO}_+(2,1)^{n-2+2g})$ of the variables $M_3, \ldots, B_g$. It follows directly from the block form of $(D|_\Sigma)^{-1}$ that $\{g, f\}_D = 0$ and hence $\{\psi, f\}_D = 0$ by (15). For $X \in \{M_3, \ldots, B_g\}$, we have $\{j_X^a, C_{i+9}\} = 0$ for all $i \in \{1, 2, 3\}$ and thus

$$\{j_X^a, g\}_D = -(\mathbb{1} - \mathrm{Ad}(u_X^{-1}))^a{}_c \sum_{j,k=1}^{3} W^c{}_{k-1}\left[(B^{-1})_{kj}\{g, C_{j+3}\} - (B^{-1}CA^{-1})_{kj}\{g, C_j\}\right]|_\Sigma,$$

where $W : \mathbb{R} \to \mathrm{Mat}(3, \mathbb{R})$ is a function of the variable ψ from (15) defined by the condition $\{j_X^a, C_{i+6}\} = -(\mathbb{1} - \mathrm{Ad}(u_X^{-1}))^a{}_b W^b{}_{i-1}$ for all $i = 1, 2, 3$, $X \in \{M_3, \ldots, B_g\}$. From equations (14) for the constraints and the definition of the matrices A, B, C it follows that the right-hand side of this equation can be expressed as a function of ψ and the fixed parameters that characterise the conjugacy classes $\mathcal{C}_1, \mathcal{C}_2$. This implies that there is a map $\boldsymbol{q}_\psi : \mathbb{R}^2 \to \mathbb{R}^3$ with $\partial_\alpha \boldsymbol{q}_\psi = 0$ such that

$$\{\psi, \boldsymbol{j}_X\}_D = -(\mathbb{1} - \mathrm{Ad}(u_X^{-1}))\boldsymbol{q}_\psi. \tag{17}$$

Similarly, we obtain for $i \in \{1, 2\}$ and functions $f \in C^\infty(\mathrm{SO}_+(2,1)^{n-2+2g})$:

$$\begin{aligned}\{j_{M_i}^a, f\}_D = \sum_{Y \in \{M_3, \ldots, B_g\}} (J_Y^{R,c} f + J_Y^{L,c} f)\Big\{ &-(\mathbb{1} - \mathrm{Ad}(u_{M_i}^{-1}))^a{}_c + \sum_{k,j=1}^{3}\Big[\{j_{M_i}^a, C_{k+9}\}(A^{-1})_{kj}\delta_c^{j-1} \\ + \{j_{M_i}^a, C_{k+6}\}(B^{-1})_{kj}\sum_{l=1}^{2}\Theta_d^{M_l, j}(\mathbb{1} - \mathrm{Ad}(u_{M_l}^{-1}))^d{}_c &- \{j_{M_i}^a, C_{k+6}\}(B^{-1}CA^{-1})_{kj}\delta_c^{j-1}\Big]\Big\}|_\Sigma. \end{aligned}$$

The term inside the curly brackets on the right-hand side again depends on ψ only, which shows that there is a map $\boldsymbol{q}_\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ with $\partial_\alpha \boldsymbol{q}_\alpha = 0$ such that

$$\{\alpha, f\}_D = \sum_{Y \in \{M_3, \ldots, B_g\}} q_\alpha^a (J_{R,a}^Y + J_{L,a}^Y) f|_\Sigma \tag{18}$$

The remaining brackets which involve ψ, α and the variables j_X^a, $X \in \{M_3, \ldots, B_g\}$, are

obtained from the Dirac brackets of $j^a_{M_i}$, $i=1,2$, with j^a_X:

$$\begin{aligned}\{j^a_{M_i}, j^b_X\}_D = -\varepsilon^{bc}{}_d j^d_X \Big[&-(\mathbb{1}-\mathrm{Ad}(u^{-1}_{M_i}))^a{}_c + \sum_{k=7}^{12}\sum_{j=1}^{3}\{j^a_{M_i}, C_k\}(D^{-1})_{kj}\delta^{j-1}_c \\ &+ \sum_{k=7}^{12}\sum_{j=4}^{6}\{j^a_{M_i}, C_k\}(D^{-1})_{kj}\sum_{l=1}^{2}\Theta^{M_l, j-4}_e(\mathbb{1}-\mathrm{Ad}(u^{-1}_{M_l}))^e{}_c\Big]|_\Sigma \\ - (\mathbb{1}-\mathrm{Ad}(u^{-1}_X))^b{}_c\Big[&\sum_{k=1}^{6}\sum_{j=7}^{9}\{j^a_{M_i}, C_k\}(D^{-1})_{kj}W^c{}_{j-7} - \sum_{k=7}^{12}\sum_{j=7}^{9}\{j^a_{M_i}, C_k\}(D^{-1})_{kj}W^c{}_{j-7}\Big]|_\Sigma.\end{aligned}$$

The term in the second set of square brackets depends on ψ and α while the term in the first set of square brackets coincides with the term in the curly brackets in the expression for $\{j^a_{M_i}, f\}_D$. This implies that there is a map $\boldsymbol{q_\theta}:\mathbb{R}^2\to\mathbb{R}^3$, $\partial^2_\alpha \boldsymbol{q_\theta}=0$ such that for all $X\in\{M_3,\ldots,B_g\}$:

$$\{\alpha, \boldsymbol{j}_X\}_D = -(\mathbb{1}-\mathrm{Ad}(u^{-1}_X))\boldsymbol{q_\theta} - \boldsymbol{q}_\alpha \wedge \boldsymbol{j}_X.$$

It remains to show that $\{\alpha,\psi\}_D=0$ and that $\boldsymbol{q}_\psi\wedge\boldsymbol{q}_\alpha=0$. With the definitions

$$\begin{aligned}(u^{-1}_{12}, \boldsymbol{j}_{12}) &\coloneqq M_1^{-1}\cdot M_2^{-1},\\ (u^{-1}_R, \boldsymbol{j}_R) &\coloneqq M_3^{-1}\cdots M_n^{-1}[A_1^{-1}, B_1]\cdots[A_g^{-1}, B_g],\end{aligned}$$

the constraints C_7, C_8, C_9 imply $\mathrm{Tr}(u_{12})=\mathrm{Tr}(u_R)$. From the Dirac brackets (18) of α with functions $f\in C^\infty(\mathrm{SO}_+(2,1)^{n-2+2g})$ of the holonomies $M_3,\ldots,B_g$, it follows that the Dirac bracket of α and ψ vanishes. Moreover, the constraint functions C_1, C_2, C_3 imply $\boldsymbol{j}_{12}=-\mathrm{Ad}(u^{-1}_{12})\boldsymbol{j}_R$ on Σ and it follows from (17) that

$$0=\{\psi, \boldsymbol{j}_{12}\}_D = -\mathrm{Ad}(u^{-1}_{12})\{\psi, \boldsymbol{j}_R\}_D = -(\mathbb{1}-\mathrm{Ad}(u^{-1}_{12}))\boldsymbol{q}_\psi. \tag{19}$$

For each function $g\in C^\infty(\mathrm{SO}_+(2,1))$, we have two associated functions $g_{\mathbb{R}^2}\in C^\infty(\mathbb{R}^2)$, $g_{\mathbb{R}^2}(\psi,\alpha)\coloneqq g(u^{-1}_{12})$ and $\bar g\in C^\infty(\mathrm{SO}_+(2,1)^{n-2+2g})$, $\bar g(u_{M_3},\ldots,u_{B_g})\coloneqq g(u_R)$. With the identity $\{\psi,\alpha\}_D=0$, we obtain

$$0=\{\alpha, g_{\mathbb{R}^2}\}_D = \{\alpha, \bar g\}_D = \sum_{Y\in\{M_3,\ldots,B_g\}} q^a_\alpha (J^Y_{R,a}+J^Y_{L,a})\bar g.$$

Together with (19), this implies that both, $\exp(q^a_\alpha J_a)$ and $\exp(q^a_\psi J_a)$, stabilise u_{12} and hence $\boldsymbol{q}_\psi\wedge\boldsymbol{q}_\alpha=0$.

3. To prove the second part of the theorem, we explicitly compute the Dirac brackets of the variables $\boldsymbol{j}_X$ for $X\in\{M_3,\ldots,B_g\}$ and functions $f\in C^\infty(\mathrm{SO}_+(2,1)^{n-2+2g})$ from expressions (16). To determine the brackets $\{\boldsymbol{j}_X, f\}_D$, we note that $\{j^a_X, C_{i+9}\}=0$ for all $i\in\{1,2,3\}$, which implies

$$\{j^a_X, f\}_D = \{j^a_X, f\}|_\Sigma - (\mathbb{1}-\mathrm{Ad}(u^{-1}_X))^{ae}\, V_{ed}\sum_{Y\in\{M_3,\ldots,B_g\}}(J^{R,d}_Y + J^{L,d}_Y)f|_\Sigma \quad \text{with} \tag{20}$$

$$V_{ed}\coloneqq W_e{}^f(B^{-1}CA^{-1})_{f+1,d+1}|_\Sigma - W_e{}^f\sum_{j=1}^{3}(B^{-1})_{f+1,j}\sum_{i=1}^{2}\Theta^{M_i,j}_a(\mathbb{1}-\mathrm{Ad}(u_{M_i}))_d{}^a|_\Sigma.$$

As none of the terms in the expression for V depend on α, it gives rise to a map $V : \mathbb{R}^2 \to \mathrm{Mat}(3, \mathbb{R})$ that satisfies $\partial_\alpha V = 0$. Similarly, we obtain

$$\{j^a_X, j^b_Y\}_D = \{j^a_X, j^b_Y\}|_\Sigma + \left(\mathbb{1} - \mathrm{Ad}(u_X^{-1})\right)^{ad} V_{dg}\, \varepsilon^{gb}{}_f j^f_Y|_\Sigma - \left(\mathbb{1} - \mathrm{Ad}(u_Y^{-1})\right)^{bd} V_{dg}\, \varepsilon^{ga}{}_f j^f_X|_\Sigma + \left(\mathbb{1} - \mathrm{Ad}(u_X^{-1})\right)^{ac}\left(\mathbb{1} - \mathrm{Ad}(u_Y^{-1})\right)^{bd} U_{cd}|_\Sigma \tag{21}$$

with $U_{cd} := W_c{}^e W_d{}^f (D^{-1})_{e+7,f+7}$ for all $c, d \in \{0, 1, 2\}$. The matrix U depends only on the parameters ψ and α, and its dependence on α is at most linear. Moreover, it follows directly from the definition of the matrix D that U is antisymmetric. This allows us to expand U in a basis: $U^{ab} = \varepsilon^{abc} m_c$ with $\boldsymbol{m} : \mathbb{R}^2 \to \mathbb{R}^3$, $\partial^2_\alpha \boldsymbol{m} = 0$.

4. By inserting the expressions (4), (5) for the left- and right-invariant vector fields on $\mathrm{ISO}(2,1)$ into the Poisson bivector (12) together with the expression for $r(\psi, \alpha)$, one obtains after some computations expressions (20), (21). This proves the claim.

□

Theorem 4.5 gives explicit expressions for the Dirac bracket for a rather general set of gauge fixing conditions. This generalises the results from [28], which investigates specific gauge fixing conditions of this type. Given the fact that the Dirac bracket is obtained from six first-class constraints with six associated gauge fixing conditions and hence involves inverting a (12×12)-Dirac matrix, its structure is surprisingly simple. This is partly due to the restriction that the gauge fixing conditions are adapted to the tangent bundle structure of $\mathrm{ISO}(2,1) = T\,\mathrm{SO}_+(2,1)$.

4.2 The Dirac bracket and the classical dynamical Yang-Baxter equation

The Dirac bracket in Theorem 4.5 defines a Poisson structure on the constraint surface $\Sigma = C^{-1}(0)$ which can be identified with a subset of $\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$. However, this identification is implicit, and it is cumbersome to give an explicit parametrisation of this subset for general gauge fixing conditions. For this reason, we consider in the following the bracket on $\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$ defined by the expressions in Theorem 4.5.

Definition 4.6. *We denote by* $\{\,,\,\}_D$ *the antisymmetric bilinear function* $\{\,,\,\}_D : C^\infty(\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}) \times C^\infty(\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}) \to C^\infty(\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g})$ *that takes the form described in Theorem 4.5. With* $\mathbb{R}^2$ *parametrised by* ψ, α *and the different copies of* $\mathrm{ISO}(2,1)$ *labelled by* $\{M_3, \ldots, M_n, A_1, B_1, \ldots, A_g, B_g\}$*, this bracket is given by:*

1. $\{\psi, \alpha\}_D = 0$*, and for all* $f \in C^\infty(\mathrm{SO}_+(2,1)^{n-2+2g})$*,* $X, Y \in \{M_3, \ldots, B_g\}$*:*

$$\begin{aligned} &\{\psi, f\}_D = 0, && \{\psi, \boldsymbol{j}_X\}_D = -(\mathbb{1} - \mathrm{Ad}(u_X^{-1}))\, \boldsymbol{q}_\psi, \\ &\{\alpha, f\}_D = \sum_{Y \in \{M_3, \ldots, B_g\}} q^a_\alpha (J^Y_{R,a} + J^Y_{L,a}) f, && \{\alpha, \boldsymbol{j}_X\}_D = -(\mathbb{1} - \mathrm{Ad}(u_X^{-1}))\boldsymbol{q}_\theta - \boldsymbol{q}_\alpha \wedge \boldsymbol{j}_X, \end{aligned} \tag{22}$$

with $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta : \mathbb{R}^2 \to \mathbb{R}^3$ *satisfying* $\boldsymbol{q}_\psi \wedge \boldsymbol{q}_\alpha = 0$ *and* $\partial_\alpha \boldsymbol{q}_\psi = \partial_\alpha \boldsymbol{q}_\alpha = \partial^2_\alpha \boldsymbol{q}_\theta = 0$.

2. *For all functions* $F, G \in C^\infty(\mathrm{ISO}(2,1)^{n-2+2g})$*:* $\{F, G\}_D = B^{n-2,g}_r(\mathrm{d}F \otimes \mathrm{d}G)$*, where* $B^{n-2,g}_r$ *is the Poisson bivector* (12) *and* $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ *is given by*

$$r(\psi, \alpha) = P_a \otimes J^a - V^{bc}(\psi)(P_b \otimes J_c - J_c \otimes P_b) + \varepsilon^{bcd} m_d(\psi, \alpha) P_b \otimes P_c, \tag{23}$$

with a map $V : \mathbb{R} \to \mathrm{Mat}(3, \mathbb{R})$ that does not depend on α and a vector-valued function $\boldsymbol{m} : \mathbb{R}^2 \to \mathbb{R}^3$ satisfying $\partial_\alpha^2 \boldsymbol{m} = 0$.

This bracket has a particularly simple structure. The two variables ψ and α Poisson-commute, and their Dirac brackets with functions on $\mathrm{ISO}(2,1)^{n-2+2g}$ are given by three functions $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta : \mathbb{R}^2 \to \mathbb{R}^3$. The Dirac bracket of two functions on $\mathrm{ISO}(2,1)^{n-2+2g}$ is again given by the Poisson bivector (12). The only difference is that the classical r-matrix $r = P_a \otimes J^a$ in the Poisson bivector $B_r^{n-2,g}$ is now replaced by the map $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ that depends on the variables ψ and α.

Note that it is a priori not guaranteed that the bracket $\{\,,\,\}_D$ on $\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$ satisfies the Jacobi identity. The Dirac gauge fixing formalism only guarantees that this is the case on the constraint surface $\Sigma = C^{-1}(0) \subset \mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$. Moreover, it is natural to ask how the Jacobi identity is encoded in the structures that characterise the bracket in Definition 4.6: the map $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ and the vector-valued functions $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta : \mathbb{R}^2 \to \mathbb{R}^3$. As the classical Yang-Baxter equation for the r-matrix in the Poisson bivector (12) ensures that the associated bracket satisfies the Jacobi identity, it is natural to expect that the Jacobi identity for the Dirac bracket follows from an analogous property of the map r. This suggests that r should be related to solutions of the classical *dynamical* Yang-Baxter equation and hence to classical dynamical r-matrices for the Lie algebra $\mathfrak{iso}(2,1)$.

This intuition is also supported by the fact that Fock and Rosly's Poisson structure is related to certain Poisson structures from the theory of Poisson-Lie groups [22, 5]. It is shown in [21, 20] that Dirac gauge fixing in the context of Poisson-Lie groupoids is linked to classical dynamical r-matrices. Note, however, that our case is more involved. While the references [21, 20] consider a gauge fixing procedure for a generalisation of the Sklyanin bracket in the context of Poisson-Lie groupoids, our Poisson structure involves several copies of the dual Poisson-Lie structure and the Heisenberg double Poisson structure which interact in a non-trivial way. Moreover, the gauge fixing conditions we consider are associated with two punctures and hence with two non-Poisson-commuting dual Poisson-Lie structures whose Poisson brackets with the remaining punctures and handles do not vanish. Nevertheless, it is natural to expect that our gauge fixing procedure should be related to solutions of the classical dynamical Yang-Baxter equation.

The concept of a classical dynamical r-matrix generalises the notion of classical r-matrices $r \in \mathfrak{g} \otimes \mathfrak{g}$ for a Lie algebra $\mathfrak{g}$ to maps $r : U \to \mathfrak{g} \otimes \mathfrak{g}$ that depend non-trivially on variables in U. The domain U is an open subset of the dual $\mathfrak{h}^*$ of an abelian Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and the map r is required to be invariant under the action of $\mathfrak{h}$. Instead of the classical Yang-Baxter equation (CYBE), the classical dynamical r-matrix is required to satisfy the classical dynamical Yang-Baxter equation (CDYBE). The latter is obtained by replacing the right-hand side of the CYBE by a term that contains the derivatives of r with respect to the coordinates on U.

Definition 4.7 ([19])**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ an abelian Lie subalgebra, and $U \subset \mathfrak{h}^*$ an open subset. A* ***classical dynamical r-matrix*** *for $(\mathfrak{g}, \mathfrak{h}, U)$ is an $\mathfrak{h}$-invariant, meromorphic function $r : U \to \mathfrak{g} \otimes \mathfrak{g}$ that satisfies the* ***classical dynamical Yang-Baxter equation*** *(CDYBE):*

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \sum_{i=1}^{dim\,\mathfrak{h}} \left(x_i^{(1)} \partial_{x^i} r_{23} - x_i^{(2)} \partial_{x^i} r_{13} + x_i^{(3)} \partial_{x^i} r_{12} \right), \quad (24)$$

where $\{x_i\}_{i=1,\dots,dim\,\mathfrak{h}}$ *is a basis of* $\mathfrak{h}$ *and* $\{x^i\}_{i=1,\dots,dim\,\mathfrak{h}}$ *the associated dual basis of* $\mathfrak{h}^*$.

In the following, we only require the case where $\mathfrak{g} = \mathfrak{iso}(2,1)$ and $\mathfrak{h}$ is a two-dimensional abelian Lie subalgebra of $\mathfrak{g}$. We thus identify $\mathfrak{h}^*$ with $\mathbb{R}^2$ and parametrise it by two variables $x^1 = \psi$, $x^2 = \alpha$. Moreover, we temporarily drop the requirements that the elements x_1, x_2 in the CDYBE form a fixed basis of $\mathfrak{h} \subset \mathfrak{iso}(2,1)$ and that r is invariant under the action of $\mathfrak{h}$.

Instead, we investigate solutions of the CDYBE (24) associated with maps $x_1, x_2 : \mathbb{R}^2 \to \mathfrak{iso}(2,1)$ of the form $x_1 = q^a_\psi P_a$ and $x_2 = q^a_\alpha J_a + q^a_\theta P_a$ with $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha : \mathbb{R} \to \mathbb{R}^3$, $\boldsymbol{q}_\theta : \mathbb{R}^2 \to \mathbb{R}^3$ satisfying $\boldsymbol{q}_\psi \wedge \boldsymbol{q}_\alpha = 0$. Note that this implies that $\mathfrak{h}(\psi,\alpha) = \mathrm{span}\{x_1(\psi,\alpha), x_2(\psi,\alpha)\}$ is a two-dimensional abelian Lie subalgebra of $\mathfrak{iso}(2,1)$ for all values of ψ and α. It is a Cartan subalgebra if and only if $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha$ satisfy the additional requirement $\boldsymbol{q}^2_\alpha, \boldsymbol{q}^2_\psi \neq 0$. We do not assume that $r(\psi,\alpha)$ is invariant under the subalgebra $\mathfrak{h}(\psi,\alpha)$.

Although such solutions of the CDYBE (24) do not correspond to classical dynamical r-matrices in the sense of Definition 4.7, admitting such generalised solutions allows us to apply the CDYBE to the maps r and $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta$ in Definition 4.6 and to determine under which conditions they give rise to a solution of the CDYBE. By comparing these conditions to the requirement that the bracket $\{\,,\}_D$ in Definition 4.6 satisfies the Jacobi identity, we obtain the following theorem.

Theorem 4.8. *The bracket* $\{\,,\}_D$ *in Definition 4.6 satisfies the Jacobi identity and hence defines a Poisson structure on* $\mathrm{ISO}(2,1)^{n-2+2g}$ *if and only if:*

1. *The map* $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ *in* (23) *satisfies the CDYBE with* $x^1 = \psi$, $x^2 = \alpha$ *and* $x_1 = q^a_\psi P_a$, $x_2 = q^a_\alpha J_a + q^a_\theta P_a$.
2. *The following additional conditions hold:*

$$\left.\begin{aligned}
0 &= q^a_\psi + \varepsilon^a{}_{bc} q^b_\psi \partial_\psi q^c_\psi + q^b_\psi V_b{}^a - q^a_\psi V^b{}_b\,,\\
0 &= \varepsilon^a{}_{dh} q^d_\alpha V^{bh} + \varepsilon^b{}_{dh} q^d_\alpha V^{ah} + \varepsilon_{cde} q^c_\alpha V^{de} \eta^{ab} - \varepsilon^b{}_{de} q^a_\alpha V^{de} + q^a_\alpha \partial_\alpha q^b_\theta - q^b_\psi \partial_\psi q^a_\alpha,\\
0 &= q^a_\theta + \varepsilon^a{}_{bc} q^b_\theta \partial_\alpha q^c_\theta + \varepsilon^a{}_{bc} q^b_\psi \partial_\psi q^c_\theta - \varepsilon^a{}_{bc} m^b q^c_\alpha + q^d_\theta V_d{}^a - q^a_\theta V_d{}^d.
\end{aligned}\right\} \quad (25)$$

Proof.

1. As a first step, we show that a map $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ of the form (23) is a solution of the CDYBE with $x^1 = \psi$, $x^2 = \alpha$, $x_1 = q^a_\psi P_a : \mathbb{R} \to \mathbb{R}^3$, $x_2 = q^a_\alpha J_a + q^a_\theta P_a : \mathbb{R}^2 \to \mathbb{R}^3$ if and only if it satisfies the equations

$$\left.\begin{aligned}
0 = \Upsilon^{abc} &:= q^a_\alpha \varepsilon^{bcd} \partial_\alpha m_d - q^b_\psi \partial_\psi V^{ca} + q^c_\psi \partial_\psi V^{ba}\\
&\qquad - V^{bd} V^{cg} \varepsilon_{dg}{}^a - V^{da} V^{cg} \varepsilon_{dg}{}^b + V^{da} V^{bg} \varepsilon_{dg}{}^c - V^{da} \varepsilon^{bc}{}_d,\\
0 = \Omega &:= \boldsymbol{q}_\psi \cdot \partial_\psi \boldsymbol{m} + \boldsymbol{q}_\theta \cdot \partial_\alpha \boldsymbol{m} + \boldsymbol{w} \cdot \boldsymbol{m} \qquad \text{with} \quad \varepsilon^{abc} w_c = V^{ab} - V^{ba}.
\end{aligned}\right\} \quad (26)$$

   Inserting expression (23) for r into the left-hand side of the CDYBE (24) and using expressions (1) for the Lie bracket of $\mathfrak{iso}(2,1)$, we obtain

$$[[r,r]] = -\boldsymbol{w} \cdot \boldsymbol{m}\, \varepsilon^{abc} P_a \otimes P_b \otimes P_c +$$
$$\big[V^{bd} V^{cg} \varepsilon_{dg}{}^a + V^{da} V^{cg} \varepsilon_{dg}{}^b - V^{da} V^{bg} \varepsilon_{dg}{}^c + V^{da} \varepsilon^{bc}{}_d\big]\,(J_a \otimes P_b \otimes P_c - P_b \otimes J_a \otimes P_c + P_b \otimes P_c \otimes J_a).$$

Setting $x^1 = \psi$, $x^2 = \alpha$, $x_1 = q_\psi^a P_a$, $x_2 = q_\alpha^a J_a + q_\theta^a P_a$ and using $\partial_\alpha V = 0$, we find that the right-hand side of the CDYBE is given by:

$$\sum_{i=1}^{2} x_i^{(1)} \partial_{x_i} r_{23} - x_i^{(2)} \partial_{x_i} r_{13} + x_i^{(3)} \partial_{x_i} r_{12} = (\boldsymbol{q}_\psi \cdot \partial_\psi \boldsymbol{m} + \boldsymbol{q}_\theta \cdot \partial_\alpha \boldsymbol{m}) \varepsilon^{abc} P_a \otimes P_b \otimes P_c +$$
$$\left(q_\alpha^a \varepsilon^{bcd} \partial_\alpha m_d - q_\psi^b \partial_\psi V^{ca} + q_\psi^c \partial_\psi V^{ba} \right) (J_a \otimes P_b \otimes P_c - P_b \otimes J_a \otimes P_c + P_b \otimes P_c \otimes J_a).$$

A comparison of the coefficients in these two expressions then yields equations (26).

2. To determine under which conditions the bracket in Definition 4.6 satisfies the Jacobi identity, we consider the variables ψ, α , functions $h \in C^\infty(\mathrm{SO}_+(2,1)^{n-2+2g})$ and the variables j_X^a for $X \in \{M_3, \ldots, B_g\}$. The structure of the Poisson algebra in Theorem 4.5 allows us to reduce the proof to six cases which are distinguished by the number of variables j_X^a, ψ, α in the brackets.

   (a) For cyclic sums over brackets of the form $\{h, \{j_Y^b, j_Z^c\}_D\}_D$ with $Y, Z \in \{M_3, \ldots, B_g\}$, we obtain

   $$\begin{aligned}\{h, \{j_Y^b, j_Z^c\}_D\}_D + \{j_Y^b, \{j_Z^c, h\}_D\}_D + \{j_Z^c, \{h, j_Y^b\}_D\}_D \\ = \left(\mathbb{1} - \mathrm{Ad}(u_Y^{-1})\right)^b{}_d \left(\mathbb{1} - \mathrm{Ad}(u_Z^{-1})\right)^c{}_e \Upsilon^{deg} \sum_{Y \in \{M_3, \ldots, B_g\}} (R_Y^g + L_Y^g) h,\end{aligned}$$

   where Υ^{deg} is the term in the first equation of (26). Consequently, it vanishes if r satisfies the CDYBE.

   (b) For cyclic sums over brackets of the form $\{j_X^a, \{j_Y^b, j_Z^c\}_D\}_D$ with $X, Y, Z \in \{M_3, \ldots, B_g\}$, we have

   $$\begin{aligned}&\{j_X^a, \{j_Y^b, j_Z^c\}_D\}_D + \{j_Y^b, \{j_Z^c, j_X^a\}_D\}_D + \{j_Z^c, \{j_X^a, j_Y^b\}_D\}_D \\ &= \left(\mathbb{1} - \mathrm{Ad}(u_Y^{-1})\right)^b{}_f \left(\mathbb{1} - \mathrm{Ad}(u_Z^{-1})\right)^c{}_e \varepsilon^a{}_{dg}\, j_X^d \Upsilon^{efg} \\ &+ \left(\mathbb{1} - \mathrm{Ad}(u_X^{-1})\right)^a{}_e \left(\mathbb{1} - \mathrm{Ad}(u_Z^{-1})\right)^c{}_f \varepsilon^b{}_{dg}\, j_Y^d \Upsilon^{efg} \\ &+ \left(\mathbb{1} - \mathrm{Ad}(u_X^{-1})\right)^a{}_f \left(\mathbb{1} - \mathrm{Ad}(u_Y^{-1})\right)^b{}_e \varepsilon^c{}_{dg}\, j_Z^d \Upsilon^{efg} \\ &+ \left(\mathbb{1} - \mathrm{Ad}(u_X^{-1})\right)^a{}_d \left(\mathbb{1} - \mathrm{Ad}(u_Y^{-1})\right)^b{}_e \left(\mathbb{1} - \mathrm{Ad}(u_Z^{-1})\right)^c{}_f \varepsilon^{def} \Omega,\end{aligned}$$

   where Υ^{efg} and Ω are, respectively, the terms in the first and second lines of (26). This shows that the Jacobi identity for brackets of this type is satisfied if and only if r is a solution of the CDYBE.

   (c) The remaining cases involve cyclic sums over brackets of the form $\{\psi, \{j_X^a . j_Y^b\}_D\}_D$, $\{\psi, \{\alpha, j_X^a\}_D\}_D$, $\{\alpha, \{h, j_Y^b\}_D\}_D$ and $\{\alpha, \{j_X^a, j_Y^b\}_D\}_D$ with $X, Y \in \{M_3, \ldots, B_g\}$. A direct calculation along the same lines as in cases (a) and (b) shows that the Jacobi identity is satisfied for brackets of this type if and only if the identities in (25) hold.

□

Theorem 4.8 gives a direct link between Poisson structures on $\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$ of the form in Definition 4.6 and solutions of the CDYBE. As is apparent in the proof, the CDYBE

is a necessary and sufficient condition which ensures that the Poisson brackets of functions $F, G \in C^\infty(\mathrm{ISO}(2,1)^{n-2+2g})$ satisfy the Jacobi identity for all values of ψ and α. The additional conditions (25) ensure that the Jacobi identity also holds for mixed brackets involving the variables ψ, α as well as functions $F \in C^\infty(\mathrm{ISO}(2,1)^{n-2+2g})$. We will show in the next section that these conditions have a direct geometrical interpretation. They allow one to locally transform a solution $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ of the CDYBE into a classical dynamical r-matrix in the sense of Definition 4.7 that is invariant under a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{iso}(2,1)$.

4.3 Examples of solutions

The conditions (26) that characterise the classical dynamical Yang-Baxter equation and the supplementary conditions (25) in Theorem 4.8 are quite complicated. It is therefore not obvious to determine solutions of these equations. In the following, we show that the specific gauge fixing conditions investigated in [28] give rise to a solution of the CDYBE that also satisfies the additional conditions (25) in Theorem 4.8. We also determine a simplified standard set of solutions of these equations that are classical dynamical r-matrices in the sense of Definition 4.7.

The publication [28] investigates a specific set of gauge fixing conditions of the type discussed in Section 3.5, which is motivated by its direct physical interpretation in the application to the Chern-Simons formulation of (2+1)-gravity. These gauge fixing conditions consider the case where the two gauge-fixed holonomies M_1, M_2 are restricted to conjugacy classes

$$\mathcal{C}_j = \{h \cdot \exp(-\mu_j J_0 - s_j P_0) \cdot h^{-1} \mid h \in \mathrm{ISO}(2,1)\} \qquad \forall j = 1, 2,$$

with $\mu_1, \mu_2 \in (0, 2\pi)$, $s_1, s_2 \in \mathbb{R}$. The resulting Dirac bracket is determined in [28]. It takes the form of Theorem 4.5 and Definition 4.6 with $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ given by

$$\left.\begin{aligned} &r(\psi,\alpha) = \tfrac{1}{2}\,(P_a \otimes J^a + J^a \otimes P_a) - \tfrac{1}{2}\varepsilon^{abc} w_c(\psi)(P_a \otimes J_b - J_b \otimes P_a) + \varepsilon^{abc} m_c(\psi,\alpha) P_a \otimes P_b, \\ &\boldsymbol{w}(\psi) = \cot\tfrac{\mu_1}{2}\,\boldsymbol{e}_0 + \cot\tfrac{\mu_1}{2}\coth\psi\,\boldsymbol{e}_1 - \coth\psi\,\boldsymbol{e}_2, \\ &\boldsymbol{m}(\psi,\alpha) = s_1/(4\sin^2\tfrac{\mu_1}{2})\,\boldsymbol{e}_0 + s_1\coth\psi/(4\sin^2\tfrac{\mu_1}{2})\,\boldsymbol{e}_1 + \tfrac{1}{2}\alpha\,\partial_\psi \boldsymbol{w}(\psi). \end{aligned}\right\} \tag{27}$$

The associated maps $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta : \mathbb{R}^2 \to \mathbb{R}^3$ take the form

$$\left.\begin{aligned} &\boldsymbol{q}_\psi(\psi,\alpha) = -\boldsymbol{q}_\alpha(\psi,\alpha) = \tfrac{1}{2}(\coth\psi\cot\tfrac{\mu_1}{2} + \cot\tfrac{\mu_2}{2}/\sinh\psi)\,\boldsymbol{e}_0 + \tfrac{1}{2}\cot\tfrac{\mu_1}{2}\,\boldsymbol{e}_1 - \tfrac{1}{2}\,\boldsymbol{e}_2, \\ &\boldsymbol{q}_\theta(\psi,\alpha) = \left[\frac{s_1\coth\psi}{4\sin^2\frac{\mu_1}{2}} + \frac{s_2}{4\sin^2\frac{\mu_2}{2}\sinh\psi} - \frac{\alpha(\cot\frac{\mu_1}{2} + \cosh\psi\cot\frac{\mu_2}{2})}{2\sinh^2\psi}\right]\boldsymbol{e}_0 + \frac{s_1}{4\sin^2\frac{\mu_1}{2}}\,\boldsymbol{e}_1. \end{aligned}\right\} \tag{28}$$

We will now show that this defines a solution of the CDYBE which also satisfies the additional conditions (25).

Lemma 4.9. *The map* $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ *in* (27) *is a solution of the CDYBE with* $x^1 = \psi$, $x^2 = \alpha$, $x_1 = q^a_\psi P_a$, $x_2 = q^a_\alpha J_a + q^a_\theta P_a$, $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta$ *as in* (28), *and satisfies the additional conditions* (25). *The maps* $x_1, x_2 : \mathbb{R}^2 \to \mathfrak{iso}(2,1)$ *define a Cartan subalgebra* $\mathfrak{h}(\psi,\alpha)$ *for all values of* ψ *for which* $\boldsymbol{q}^2_\psi(\psi), \boldsymbol{q}^2_\alpha(\psi) \neq 0$.

Proof.

1. That the maps $x_1, x_2 : \mathbb{R}^2 \to \mathfrak{iso}(2,1)$ define a two-dimensional abelian Lie subalgebra of $\mathfrak{iso}(2,1)$ follows directly from the condition $\boldsymbol{q}_\psi \wedge \boldsymbol{q}_\alpha = 0$. One finds that this Lie subalgebra is a Cartan subalgebra for all values of ψ for which $\boldsymbol{q}_\psi^2(\psi) \neq 0$.

2. That r solves the CDYBE can be shown with a direct calculation. Inserting the expressions (27) for $\boldsymbol{w}, \boldsymbol{m}$ and expressions (28) for $\boldsymbol{q}_\psi, \boldsymbol{q}_\theta$ into the left-hand-side of the second equation in (26), one finds after some computations that this expression vanishes. To verify that the first set of equations in (26) is satisfied, we note that for maps $V : \mathbb{R} \to \mathrm{Mat}(3,\mathbb{R})$ of the form $V^{ab}(\psi) = \frac{1}{2}\eta^{ab} + \frac{1}{2}\varepsilon^{ab}{}_c w^c(\psi)$ with $\boldsymbol{w} : \mathbb{R} \to \mathbb{R}^3$, the first line of (26) is equivalent to the following conditions

$$1 + \boldsymbol{w}^2 + 2\boldsymbol{q}_\psi \cdot \partial_\psi \boldsymbol{w} = 0, \qquad \partial_\psi \boldsymbol{w} \wedge \partial_\alpha \boldsymbol{m} = 0, \qquad \partial_\alpha m^a q_\alpha^b = -\tfrac{1}{2}\partial_\psi w^a q_\psi^b. \tag{29}$$

   Setting $\boldsymbol{q}_\alpha = -\boldsymbol{q}_\psi$ and inserting expressions (27) for $\boldsymbol{w}, \boldsymbol{m}$ together with expression (28) for $\boldsymbol{q}_\psi$ into (29), one verifies the first condition in (26).

3. To show that r and $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta$ satisfy the additional conditions (25), we note that for matrices V of the form $V^{ab}(\psi) = \frac{1}{2}\eta^{ab} + \frac{1}{2}\varepsilon^{abc} w^c(\psi)$, these conditions reduce to the following set of equations

$$\left.\begin{aligned} 0 &= \boldsymbol{q}_\psi \wedge (\partial_\psi \boldsymbol{q}_\psi - \tfrac{1}{2}\boldsymbol{w}), \\ 0 &= \tfrac{1}{2}(q_\alpha^b w^a - q_\alpha^a w^b) + q_\alpha^a \partial_\alpha q_\theta^b - q_\psi^b \partial_\psi q_\alpha^a \quad \forall a,b \in \{0,1,2\}, \\ 0 &= \boldsymbol{q}_\theta \wedge (\partial_\alpha \boldsymbol{q}_\theta - \tfrac{1}{2}\boldsymbol{w}) + \boldsymbol{q}_\psi \wedge \partial_\psi \boldsymbol{q}_\theta + \boldsymbol{q}_\alpha \wedge \boldsymbol{m}. \end{aligned}\right\} \tag{30}$$

   Inserting expressions (27) for $\boldsymbol{w}, \boldsymbol{m}$ and expressions (28) for $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta$ into these equations, one finds that they are indeed satisfied.

□

Note that the resulting solution of the CDYBE in [28] is *not* a classical dynamical r-matrix in the sense of Definition 4.7. While Definition 4.7 requires the choice of an abelian Lie subalgebra $\mathfrak{h} \subset \mathfrak{iso}(2,1)$ and an identification of the two variables in the solution with its dual, the abelian Lie subalgebra $\mathfrak{h}(\psi,\alpha) = \mathrm{span}\{q_\psi^a P_a, q_\alpha^a J_a + q_\theta^a P_a\}$ associated with the above solution varies with ψ and α. A direct calculation shows that depending on the value of ψ, the Lie subalgebra $\mathfrak{h}(\psi,\alpha)$ is conjugate either to the Cartan subalgebra $\mathfrak{h}_a = \mathrm{span}\{J_0, P_0\}$ for $\boldsymbol{q}_\psi^2(\psi) > 0$, to the Cartan subalgebra $\mathfrak{h}_b = \mathrm{span}\{J_1, P_1\}$ for $\boldsymbol{q}_\psi^2(\psi) < 0$ or to the two-dimensional Lie subalgebra $\mathfrak{h}_c = \mathrm{span}\{J_0 + J_1, P_0 + P_1\}$ for $\boldsymbol{q}_\psi^2(\psi) = 0$.

The solution therefore combines solutions of the CDYBE that are associated with different, non-conjugate two-dimensional Lie subalgebras of $\mathfrak{iso}(2,1)$. To show that the existence of solutions associated with different Lie subalgebras is a generic phenomenon and not a consequence of the specific gauge fixing conditions in [28], we determine a simple set of solutions of a similar form.

Lemma 4.10. *For all $c \in \mathbb{R}$, $\gamma \in C^\infty(\mathbb{R})$, the map $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ given by*

$$r(\psi,\alpha) = \tfrac{1}{2}(P_a \otimes J^a + J^a \otimes P_a) - \varepsilon^{ab}{}_c\, \partial_\psi q_\psi^c(\psi)(P_a \otimes J_b - J_b \otimes P_a) - \alpha\, \varepsilon^{ab}{}_c\, \partial_\psi^2 q_\psi^c(\psi)\, P_a \otimes P_b$$

is a solution of the CDYBE with $x_1 = \psi$, $x_2 = \alpha$, $x^1 = q_\psi^a P_a$, $x^2 = q_\alpha^a J_a + q_\theta^a P_a$ and

$$\boldsymbol{q}_\psi(\psi) = \boldsymbol{q}_\alpha(\psi) = \gamma(\psi)\boldsymbol{e}_0 + \sqrt{\gamma^2(\psi) + \tfrac{1}{4}(\psi - c)^2}\, \boldsymbol{e}_1, \qquad \boldsymbol{q}_\theta(\psi,\alpha) = \alpha\, \partial_\psi \boldsymbol{q}_\psi(\psi).$$

Proof. This follows by a direct calculation. As r is of a form similar to the solution in Lemma 4.9 with $\boldsymbol{w} = 2\partial_\psi \boldsymbol{q}_\psi$, $\boldsymbol{m} = -\alpha\, \partial^2_\psi \boldsymbol{q}_\psi$, inserting these expressions into (30) shows directly that the conditions (25) are satisfied. The CDYBE then reduces to the requirement $1 + 2\partial^2_\psi(\boldsymbol{q}^2_\psi) = 0$, which is verified by a simple computation. □

Note, however, that the CDYBE (26) and the additional requirements (25) also admit solutions which are associated with fixed Cartan subalgebras of $\mathfrak{iso}(2,1)$ and define classical dynamical r-matrices in the sense of Definition 4.7. To obtain such solutions, we set $\boldsymbol{q}_\theta(\psi,\alpha) = 0$ and either $\boldsymbol{q}_\psi(\psi) = \boldsymbol{q}_\alpha(\psi) = \boldsymbol{e}_0$ or $\boldsymbol{q}_\psi(\psi) = \boldsymbol{q}_\alpha(\psi) = \boldsymbol{e}_1$ for all admissible values of ψ. The conditions (30) then reduce to the requirements $\boldsymbol{w}, \boldsymbol{m} \in \mathrm{span}\{\boldsymbol{q}_\psi\}$, and the expressions (29) to $\partial_\alpha \boldsymbol{m} = -\frac{1}{2}\partial_\psi \boldsymbol{w}$, $1 + \boldsymbol{w}^2 + 2\boldsymbol{q}_\psi \cdot \partial_\psi \boldsymbol{w} = 0$. From this, one then obtains two solutions associated with the Cartan subalgebras $\mathfrak{h}_a = \mathrm{span}\{J_0, P_0\}$ and $\mathfrak{h}_b = \mathrm{span}\{J_1, P_1\}$ in $\mathfrak{iso}(2,1)$.

Lemma 4.11. *Two solutions of the CDYBE with $x^1 = \psi$, $x^2 = \alpha$, $x_1 = q^a_\psi P_a$ and $x_2 = q^a_\alpha J_a + q^a_\theta P_a$ that also satisfy the additional conditions (25) are given by*

a) $\boldsymbol{q}_\psi = \boldsymbol{q}_\alpha = \boldsymbol{e}_0$, $\boldsymbol{q}_\theta = 0$ and $r : (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R} \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$,

$$r_a(\psi,\alpha) = \tfrac{1}{2}(P_a \otimes J^a + J^a \otimes P_a) + \tfrac{1}{2}\tan\tfrac{\psi}{2}\,(P_1 \wedge J_2 - P_2 \wedge J_1) + \frac{\alpha}{4\cos^2\frac{\psi}{2}} P_1 \wedge P_2,$$

b) $\boldsymbol{q}_\psi = \boldsymbol{q}_\alpha = \boldsymbol{e}_1$, $\boldsymbol{q}_\theta = 0$ and $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$,

$$r_b(\psi,\alpha) = \tfrac{1}{2}(P_a \otimes J^a + J^a \otimes P_a) + \tfrac{1}{2}\tanh\tfrac{\psi}{2}\,(P_2 \wedge J_0 - P_0 \wedge J_2) + \frac{\alpha}{4\cosh^2\frac{\psi}{2}} P_2 \wedge P_0,$$

where $X \wedge Y := X \otimes Y - Y \otimes X$. They are classical dynamical r-matrices as in Definition 4.7 for, respectively, the Cartan subalgebras $\mathfrak{h}_a = \mathrm{span}\{J_0, P_0\}$ and $\mathfrak{h}_b = \mathrm{span}\{J_1, P_1\}$.

Lemma 4.11 provides us with two particularly simple classical dynamical r-matrices for $\mathfrak{iso}(2,1)$. We will show in the next section that every solution of the CDYBE of the form (23) which satisfies the additional conditions (25) can be transformed into one of these two solutions for all values of ψ for which either $\boldsymbol{q}^2_\psi(\psi), \boldsymbol{q}^2_\alpha(\psi) > 0$ (case a) or $\boldsymbol{q}^2_\psi(\psi), \boldsymbol{q}^2_\alpha(\psi) < 0$ (case b).

One might wonder if there are similar solutions of the CDYBE and conditions (25) for which $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha$ are fixed lightlike vectors that do not depend on ψ and α. However, it turns out that such solutions do not exist. This appears to be linked to the fact that the vectors $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha$ associated with the solutions in Lemma 4.9 and Lemma 4.10 are spacelike or timelike for values of ψ in certain open intervals of $\mathbb{R}$, but can become lightlike only for a very specific discrete set of values of ψ. This again suggests that the variation of $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha$ with ψ is a generic feature of the gauge fixing procedure, and that there are no gauge fixing conditions that allow one to obtain a Poisson structure determined by vectors $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha$ that are lightlike for all ψ. We have the following lemma:

Lemma 4.12. *There are no simultaneous solutions of the CDYBE (24) and conditions (25) for which $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta$ are constant vectors with $\boldsymbol{q}_\psi \wedge \boldsymbol{q}_\alpha = 0$ and $\boldsymbol{q}^2_\psi = \boldsymbol{q}^2_\alpha = 0$.*

Proof. Suppose that $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ is of the form in Definition 4.6 with $V^{ab}(\psi) = \frac{1}{2}\eta^{ab} + Q^{ab}(\psi) + \frac{1}{2}\varepsilon^{abc} w_c(\psi)$, where $Q : \mathbb{R} \to \mathrm{Mat}(3,\mathbb{R})$ is symmetric. Then conditions (25) imply $Q^{02} = Q^{12} = Q^{22} = 0$, $Q^{00} = Q^{01} = Q^{11}$ and $\boldsymbol{w} \in \mathrm{span}\{\boldsymbol{e}_0 + \boldsymbol{e}_1\}$. Inserting this into the first equation of (26) yields a contradiction and thus proves the claim. □

5 Transformations between Dirac brackets

5.1 Dynamical Poincaré transformations

The results of the previous section show how constraint functions that satisfy the conditions in Definition 4.4 and the requirements a and b from Section 3.5 give rise to a Dirac bracket that is given by solutions of the CDYBE. In this section, we investigate the transformation of the Dirac bracket under a change of constraint functions.

For this, recall from the discussion before Theorem 4.5 that any set of admissible constraint functions restricts the variables M_1, M_2 which parametrise the first two copies of $\mathrm{ISO}(2,1)$ in $\mathrm{ISO}(2,1)^{n+2g}$ in such a way that they are determined uniquely by the two conjugation invariant quantities ψ, α which depend only on the product $M_2 \cdot M_1$. This suggests that for any two sets of variables M_1, M_2 and M_1', M_2' obtained in this way, there should be a Poincaré transformation $p(\psi,\alpha) \in \mathrm{ISO}(2,1)$ such that $M_1' = p(\psi,\alpha)\cdot M_1 \cdot p(\psi,\alpha)^{-1}$ and $M_2' = p(\psi,\alpha)\cdot M_2 \cdot p(\psi,\alpha)^{-1}$. If the associated gauge fixing conditions satisfy conditions a and b from Section 3.5, it follows from (15) that one can restrict attention to Poincaré transformations $p(\psi,\alpha)$ whose Lorentzian components do not depend on α and whose translational components depend on α at most linearly.

This is a strong motivation to investigate the transformation of the bracket in Definition 4.6 under such Poincaré transformations. We therefore consider smooth maps

$$\begin{aligned} &\Phi^p : \mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g} \to \mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}, \\ &(\psi,\alpha,M_3,\ldots,B_g) \mapsto (\psi,\alpha,\; p(\psi,\alpha)\cdot M_3\cdot p(\psi,\alpha)^{-1},\ldots,\; p(\psi,\alpha)\cdot B_g \cdot p(\psi,\alpha)^{-1}), \end{aligned} \tag{31}$$

where $p = (g, -\mathrm{Ad}(g)\boldsymbol{t}) \in C^\infty(\mathbb{R}^2, \mathrm{ISO}(2,1))$ with $\partial_\alpha g = \partial_\alpha^2 \boldsymbol{t} = 0$. We find that the transformation of the bracket $\{\,,\}_D$ under such a dynamical Poincaré transformation corresponds to a simultaneous transformation of the maps $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1)\otimes\mathfrak{iso}(2,1)$ and $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta : \mathbb{R}^2 \to \mathbb{R}^3$.

Lemma 5.1. *Let $\{,\}_D$ and $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1)\otimes\mathfrak{iso}(2,1)$ be given as in Definition 4.6 and consider a dynamical Poincaré transformation $\Phi^p : \mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g} \to \mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$ as above. Then for all $F, G \in C^\infty(\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g})$:*

$$\{F\circ\Phi^p, G\circ\Phi^p\}_D = \{F,G\}_D^p \circ \Phi^p,$$

where $\{\,,\}_D^p$ is the bracket from Definition 4.6 associated with

$$\left.\begin{aligned} &\boldsymbol{q}_\psi^p = \mathrm{Ad}(g)\boldsymbol{q}_\psi, \qquad \boldsymbol{q}_\alpha^p = \mathrm{Ad}(g)\boldsymbol{q}_\alpha, \qquad \boldsymbol{q}_\theta^p = \mathrm{Ad}(g)(\boldsymbol{q}_\theta - \boldsymbol{q}_\alpha \wedge \boldsymbol{t}), \\ &r^p = (\mathrm{Ad}(p)\otimes\mathrm{Ad}(p))\,[r + \bar\eta^p - \bar\eta^p_{21}]\,, \end{aligned}\right\} \tag{32}$$

and $\bar\eta^p : \mathbb{R}^2 \to \mathfrak{iso}(2,1)\otimes\mathfrak{iso}(2,1)$ is given by

$$\bar\eta^p = q_\psi^a P_a \otimes p^{-1}\partial_\psi p + (q_\alpha^a J_a + q_\theta^a P_a)\otimes p^{-1}\partial_\alpha p. \tag{33}$$

Proof.

1. To derive explicit expressions for the transformed bracket $\{,\}_D^p$, it is convenient to consider two cases separately, namely Lorentz transformations $g : \mathbb{R}^2 \to \mathrm{SO}_+(2,1)$, which do not depend on α, and translations $t : \mathbb{R}^2 \to \mathbb{R}^3$, which depend on α at most linearly.

We start by determining concrete expressions for r^p from formula (32). For a Lorentz transformation $p=(g,0):\mathbb{R}\to \mathrm{SO}_+(2,1)$ with $\partial_\alpha p=0$, we have $\bar{\eta}^p(\psi)=q^a_\psi(\psi)P_a\otimes g^{-1}\partial_\psi g(\psi)$. Expanding this terms of a basis as $g^{-1}\partial_\psi g(\psi)=n^a(\psi)J_a$ then yields

$$r^p=(\mathrm{Ad}(g)\otimes\mathrm{Ad}(g))[r+q^a_\psi n^b(P_a\otimes J_b-J_b\otimes P_a)]. \tag{34}$$

In the case of a translation $p=(1,\boldsymbol{t}):\mathbb{R}^2\to\mathbb{R}$ with $\partial^2_\alpha\boldsymbol{t}=0$, we have $\bar{\eta}^p=q^a_\psi\partial_\psi t^bP_a\otimes P_b+q^a_\alpha\partial_\alpha t^bJ_a\otimes P_b+q^a_\theta\partial_\alpha t^bP_a\otimes P_b$. Inserting this with expression (23) for r into (32) and using the identities $\mathrm{Ad}(\boldsymbol{t})J_a=J_a+\varepsilon_{ab}{}^c t^bP_c$, $\mathrm{Ad}(\boldsymbol{t})P_a=P_a$, we obtain after some computations

$$\begin{aligned}r^p=r&-\partial_\alpha t^aq^b_\alpha(P_a\otimes J_b-J_b\otimes P_a)\\&+\varepsilon^{ab}{}_c\big[(1-V^d{}_d)\boldsymbol{t}+V^T\boldsymbol{t}+\boldsymbol{q}_\psi\wedge\partial_\psi\boldsymbol{t}+[\boldsymbol{q}_\theta-\boldsymbol{q}_\alpha\wedge\boldsymbol{t}]\wedge\partial_\alpha\boldsymbol{t}\big]^cP_a\otimes P_b.\end{aligned}\tag{35}$$

2. We now derive explicit expressions for the transformed Poisson brackets $\{\,,\,\}^p_D$. For a Lorentz transformation $p=(g,0):\mathbb{R}\to\mathrm{SO}_+(2,1)$ with $\partial_\alpha p=0$, it follows directly from the identity $\{\psi,\alpha\}_D=0$ that the Poisson brackets involving the variables ψ and α with functions on $\mathrm{ISO}(2,1)^{n-2+2g}$ are given by:

$$\begin{aligned}&\{\psi\circ\Phi^p,h\circ\Phi^p\}_D=0,\\&\{\psi\circ\Phi^p,\boldsymbol{j}_X\circ\Phi^p\}_D=\big[-(\mathbb{1}-\mathrm{Ad}(u_X^{-1}))\,\mathrm{Ad}(g)\,\boldsymbol{q}_\psi\big]\circ\Phi^p,\\&\{\alpha\circ\Phi^p,h\circ\Phi^p\}_D=\big[\sum_{Y\in\{M_3,\dots,B_g\}}\mathrm{Ad}(g)^a{}_bq^b_\alpha(J^Y_{R,a}+J^Y_{L,a})h\big]\circ\Phi^p,\\&\{\alpha\circ\Phi^p,\boldsymbol{j}_X\circ\Phi^p\}_D=\big[-(\mathbb{1}-\mathrm{Ad}(u_X^{-1}))\,\mathrm{Ad}(g)\boldsymbol{q}_\theta-(\mathrm{Ad}(g)\boldsymbol{q}_\alpha)\wedge\boldsymbol{j}_X\big]\circ\Phi^p,\end{aligned}$$

for all $h\in C^\infty(\mathrm{SO}_+(2,1)^{n-2+2g})$ and $X\in\{M_3,\dots,B_g\}$. To determine the brackets of the type $\{F,G\}^p_D$ with $F,G\in C^\infty(\mathrm{ISO}(2,1)^{n-2+2g})$, we again consider functions $h\in C^\infty(\mathrm{SO}_+(2,1)^{n-2+2g})$ and the variables j_X, $X\in\{M_3,\dots,B_g\}$. After some straightforward computations, we obtain

$$\begin{aligned}&\{j^a_X\circ\Phi^p,h\circ\Phi^p\}_D=\\&\quad\{j^a_X,h\}\circ\Phi^p-\mathrm{Ad}(g)^a{}_c(\mathbb{1}-\mathrm{Ad}(u_X^{-1}))^c{}_h(V^h{}_e-q^h_\psi n_e)\Big[\sum_{Y\in\{M_3,\dots,B_g\}}(J^{R,e}_Y+J^{L,e}_Y)h\Big]\circ\Phi^p,\end{aligned}$$

which allows us to express the transformed bracket as

$$\begin{aligned}&\{j^a_X\circ\Phi^p,h\circ\Phi^p\}_D=\Big[\{j^a_X,h\}-(\mathbb{1}-\mathrm{Ad}(u_X^{-1}))^a{}_h(V^p)^h{}_e\sum_{Y\in\{M_3,\dots,B_g\}}(J^{R,e}_Y+J^{L,e}_Y)h\Big]\circ\Phi^p,\\&\text{with}\quad(V^p)^h{}_e=\mathrm{Ad}(g)^h{}_m\,\mathrm{Ad}(g)_e{}^p[V^m{}_p-q^m_\psi n_p].\end{aligned}$$

An analogous calculation for the brackets of the form $\{j^a_X\circ\Phi^p,j^b_Y\circ\Phi^p\}_D$ shows that it is obtained by transforming $V\to V^p$ as above and replacing $\boldsymbol{m}:\mathbb{R}^2\to\mathbb{R}^3$ by $\boldsymbol{m}^p=\mathrm{Ad}(g)\boldsymbol{m}:\mathbb{R}^2\to\mathbb{R}^3$. This implies that the transformed bracket takes the form $\{F\circ\Phi^p,G\circ\Phi^p\}_D=[B^{n-2,g}_{r^p}(\mathrm{d}F\otimes\mathrm{d}G)]\circ\Phi^p$ with r^p given by (34) and proves the claim for the Lorentz transformations.

3. To determine the transformation of the bracket $\{\,,\,\}_D$ under translations, we again use the parametrisation in terms of the variables j_X, $X\in\{M_3,\dots,B_g\}$ and functions

$h \in C^\infty(\mathrm{SO}_+(2,1)^{n-2+2g})$. The transformation of these variables under a translation $p = (0, \boldsymbol{t}) : \mathbb{R}^2 \to \mathbb{R}^3$ is given by $h \circ \Phi^p = h$ and $\boldsymbol{j}_X \circ \Phi^p = \boldsymbol{j}_X + (\mathbb{1} - \mathrm{Ad}(u_X^{-1}))\boldsymbol{t}$. This implies directly that the brackets $\{\psi, h\}_D$ and $\{\alpha, h\}_D$ are preserved, and with the relations $\{\alpha, \psi\}_D = 0$, $\{\psi, h\}_D = 0$, one obtains the same for the brackets $\{\psi, \boldsymbol{j}_X\}_D$. The formula for the brackets $\{\alpha, \boldsymbol{j}_X\}^p_D$ follows directly from the relation

$$\left\{\alpha, (\mathbb{1} - \mathrm{Ad}(u_X^{-1}))^a{}_b\right\}_D = \left[\varepsilon_d{}^a{}_m(\mathbb{1} - \mathrm{Ad}(u_X^{-1}))^m{}_b + \varepsilon_{db}{}^m(\mathbb{1} - \mathrm{Ad}(u_X^{-1}))^a{}_m\right] q^d_\alpha.$$

We thus find that the transformed brackets involving the variables ψ, α are given by

$$\begin{aligned}
&\{\psi \circ \Phi^p, h \circ \Phi^p\}_D = 0,\\
&\{\psi \circ \Phi^p, \boldsymbol{j}_X \circ \Phi^p\}_D = \left[-(\mathbb{1} - \mathrm{Ad}(u_X^{-1}))\, \boldsymbol{q}_\psi\right] \circ \Phi^p,\\
&\{\alpha \circ \Phi^p, h \circ \Phi^p\}_D = \Big[\sum_{Y \in \{M_3, \ldots, B_g\}} q^a_\alpha (J^Y_{R,a} + J^Y_{L,a}) h\Big] \circ \Phi^p,\\
&\{\alpha \circ \Phi^p, \boldsymbol{j}_X \circ \Phi^p\}_D = \left[-(\mathbb{1} - \mathrm{Ad}(u_X^{-1}))(\boldsymbol{q}_\theta - \boldsymbol{q}_\alpha \wedge \boldsymbol{t}) - \boldsymbol{q}_\alpha \wedge \boldsymbol{j}_X\right] \circ \Phi^p.
\end{aligned}$$

To determine the transformed brackets $\{F \circ \Phi^p, G \circ \Phi^p\}_D$ for $F, G \in C^\infty(\mathrm{ISO}(2,1)^{n-2+2g})$, we calculate the brackets of functions $h \in C^\infty(\mathrm{SO}_+(2,1)^{n-2+2g})$ and variables $\boldsymbol{j}_X, \boldsymbol{j}_Y$ for $X, Y \in \{M_3, \ldots, B_g\}$. A direct computation yields

$$\begin{aligned}
&\{j^a_X \circ \Phi^p, h \circ \Phi^p\}_D = \Big[\{j^a_X, h\} - (\mathbb{1} - \mathrm{Ad}(u_X^{-1}))^a{}_g (V^p)^g{}_d \sum_{Y \in \{M_3, \ldots, B_g\}} (J^{R,d}_Y + J^{L,d}_Y) h\Big] \circ \Phi^p,\\
&\{j^a_X \circ \Phi^p, j^b_Y \circ \Phi^p\}_D = \Big[\{j^a_X, j^b_Y\} + (\mathbb{1} - \mathrm{Ad}(u_X^{-1}))^a{}_g (V^p)^g{}_d\, \varepsilon^{db}{}_f j^f_Y\\
&\quad - (\mathbb{1} - \mathrm{Ad}(u_Y^{-1}))^b{}_g (V^p)^g{}_d\, \varepsilon^{da}{}_f j^f_X + (\mathbb{1} - \mathrm{Ad}(u_X^{-1}))^a{}_c (\mathbb{1} - \mathrm{Ad}(u_Y^{-1}))^b{}_d \varepsilon^{cdf} m^p_f\Big] \circ \Phi^p,\\
&\text{with } (V^p)^{bc} = V^{bc} + \partial_\alpha t^b q^c_\alpha,\\
&\text{and } \boldsymbol{m}^p = \boldsymbol{m} + (1 - V^d{}_d)\boldsymbol{t} + V^T \boldsymbol{t} + \boldsymbol{q}_\psi \wedge \partial_\psi \boldsymbol{t} + (\boldsymbol{q}_\theta - \boldsymbol{q}_\alpha \wedge \boldsymbol{t}) \wedge \partial_\alpha \boldsymbol{t}.
\end{aligned}$$

For all $F, G \in C^\infty(\mathrm{ISO}(2,1)^{n-2+2g})$ the transformed bracket therefore takes the form $\{F \circ \Phi^p, G \circ \Phi^p\}_D = [B^{n-2,g}_{r^p}(\mathrm{d}F \otimes \mathrm{d}G)] \circ \Phi^p$ with r^p given by (35). This proves the claim.

□

Lemma 5.1 gives explicit expressions for the transformation of the bracket in Definition 4.6 under dynamical Poincaré transformations which depend on the variables ψ, α and act diagonally on $\mathrm{ISO}(2,1)^{n-2+2g}$. It allows one to identify the transformed bracket with another bracket of the form in Definition 4.6 associated with transformed maps $r^p : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ and $\boldsymbol{q}^p_\psi, \boldsymbol{q}^p_\alpha, \boldsymbol{q}^p_\theta : \mathbb{R}^2 \to \mathbb{R}^3$.

As the bracket $\{\,,\,\}_D$ in Definition 4.6 is modelled after the Dirac bracket in Theorem 4.5 and Poincaré transformations of this type can be viewed as transitions between different gauge fixing conditions, it is natural to ask whether these Poincaré transformations preserve the Jacobi identity. For this, note that, given any Poisson manifold $(M, \{\,,\,\})$ and a diffeomorphism $\Phi : M \to M$, one obtains a new Poisson bracket $\{\,,\,\}^\Phi$ on M by setting $\{f, g\}^\Phi := \{f \circ \Phi, g \circ \Phi\} \circ \Phi^{-1}$. As shown in Lemma 5.1, the bracket $\{\,,\,\}^p_D = \{\,,\,\}^{\Phi^p}_D$ obtained from $\{\,,\,\}_D$ by applying the diffeomorphism Φ^p from (31) is again of the form in Definition 4.6, but with the transformed maps $r^p, \boldsymbol{q}^p_\psi, \boldsymbol{q}^p_\alpha, \boldsymbol{q}^p_\theta$. From Theorem 4.8 we thus deduce that these transformed maps are solutions of the CDYBE (24) and the additional conditions (25) if and only if the original maps $r, \boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta$ are.

Corollary 5.2. *Let $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ as in* (23) *be a solution of the CDYBE with $x^1 = \psi$, $x^2 = \alpha$, $x_1 = q^a_\psi P_a$, $x_2 = q^a_\alpha J_a + q^a_\theta P_a$ such that the conditions in* (25) *are satisfied and let $p : \mathbb{R}^2 \to \mathrm{ISO}(2,1)$ be a dynamical Poincaré transformation as in Lemma 5.1. Then $r^p : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ is a solution of the CDYBE with $x^1 = \psi$, $x^2 = \alpha$ and $x_1 = q^{p,a}_\psi P_a$, $x_2 = q^{p,a}_\alpha J_a + q^{p,a}_\theta P_a$ and satisfies* (25)*. The map Φ^p is a Poisson isomorphism between the Poisson structures $\{\,,\,\}_D$ and $\{\,,\,\}^p_D$.*

As discussed in the previous section, the equivalence of the CDYBE and conditions (25) with the Jacobi identity for the bracket in Definition 4.6 suggests that the solutions should be viewed as a generalisation of the classical dynamical r-matrices in Definition 4.7 for which the associated abelian subalgebra of $\mathfrak{iso}(2,1)$ is allowed to vary with the variables ψ and α. The transformation formula (32) for the maps $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ under dynamical Poincaré transformations and the fact that these Poincaré transformations preserve the Jacobi identity suggests that these Poincaré transformations should be interpreted as a generalised version of the gauge transformations of classical dynamical r-matrices introduced by Etingof and Varchenko in their work on the classification of classical dynamical r-matrices [19] (see also [34, 17, 18, 41] for further work on the classification). We summarise the relevant definitions and results from [19].

Definition 5.3 ([19])**.** *Let G be a Lie group, $H \subset G$ an abelian subgroup and $r : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$ a classical dynamical r-matrix for $(\mathfrak{h}, \mathfrak{g})$. A* ***gauge transformation*** *of r is a smooth function $\Pi : \mathfrak{h}^* \to G^H$ into the centraliser G^H of H in G which acts on r according to*[3]

$$r^\Pi = \big(\mathrm{Ad}(\Pi) \otimes \mathrm{Ad}(\Pi)\big)[r + \bar{\eta}^\Pi - \bar{\eta}^\Pi_{21}], \tag{36}$$

where $\bar{\eta}_\Pi : \mathfrak{h}^* \to \mathfrak{h} \otimes \mathfrak{g}^H$ is the map dual to the $\mathfrak{g}^\mathfrak{h}$-valued one-form $\eta_\Pi = \Pi^{-1}\mathrm{d}\Pi$ on $\mathfrak{h}^*$ and $\bar{\eta}^{21}_\Pi$ denotes its flip with values in $\mathfrak{g}^\mathfrak{h} \otimes \mathfrak{h}$.

The name gauge transformation is motivated by the fact that it maps classical dynamical r-matrices for $(\mathfrak{h}, \mathfrak{g})$ to classical dynamical r-matrices for $(\mathfrak{h}, \mathfrak{g})$. It is shown in [19] that if r is an $\mathfrak{h}$-invariant solution of the CDYBE, then this also holds for the transformed r-matrix r^Π.

Theorem 5.4 ([19])**.** *Let G be a Lie group, $H \subset G$ an abelian subgroup and $r : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$ a classical dynamical r-matrix for $(\mathfrak{h}, \mathfrak{g})$. Then for every gauge transformation $\Pi : \mathfrak{h}^* \to G^H$, r^Π is a classical dynamical r-matrix for $(\mathfrak{h}, \mathfrak{g})$.*

By comparing formula (32) for the action of dynamical Poincaré transformations on the solutions of the CDYBE with the one in Definition 5.3, it becomes apparent that the two expressions agree. The only difference is that in Definition 5.3 the gauge transformations are restricted to take values in the centraliser of the subgroup $H \subset G$, while no such condition is imposed in our case. Consequently, the dynamical Poincaré transformations also act on the maps $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta : \mathbb{R}^2 \to \mathbb{R}^3$ and hence on the associated two-dimensional Lie subalgebra $\mathfrak{h}(\psi, \alpha) = \mathrm{span}\{q^a_\psi P_a, q^a_\alpha J_a + q^a_\theta P_a\}$. This is also apparent in the formula for the map $\bar{\eta}^p : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ in (33), which depends on the chosen basis of the subalgebra $\mathfrak{h}(\psi, \alpha)$.

It is therefore instructive to consider the gauge transformations (36) for the classical dynamical r-matrices from Lemma 4.11 which are associated with fixed Cartan subalgebras $\mathfrak{h}_a =$

[3] The sign difference between this formula and the one in [19] is due to a different sign convention for the CDYBE (24).

span$\{P_0, J_0\} \subset \mathfrak{iso}(2,1)$ and $\mathfrak{h}_b = \text{span}\{P_1, J_1\} \subset \mathfrak{iso}(2,1)$. In that case, the abelian subgroup H in Definition 5.3 is obtained by exponentiating, respectively, the Cartan subalgebras $\mathfrak{h}_a$ and $\mathfrak{h}_b$, and the associated centraliser G^H coincides with H. With our additional restriction that the Lorentzian component of Π does not depend on α and its translational component depends on α at most linearly, the map $\Pi : \mathfrak{h}^* \to G^H$ in Definition 5.3 therefore takes the form $\Pi(\psi, \alpha) = (\exp(-\beta(\psi)J_j), -[\gamma(\psi) + \alpha\delta(\psi)]\boldsymbol{e}_j)$ with $\beta, \gamma, \delta : \mathbb{R} \to \mathbb{R}$ and $j = 0$ in case a and $j = 1$ in case b). The transformation of the classical dynamical r-matrices $r_{a,b} : \mathbb{R} \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ in Lemma 4.11 under Π is thus given by

$$r^\Pi(\psi, \alpha) = r(\psi, \alpha) - [\beta'(\psi) - \delta(\psi)]P_j \wedge J_j,$$

with $j = 0$ in case a) and $j = 1$ in case b). As $[\beta'(\psi) - \delta(\psi)]P_j \wedge J_j$ satisfies the classical dynamical Yang-Baxter equation and because $[[r(\psi, \alpha), P_j \wedge J_j]] + [[P_j \wedge J_j, r(\psi, \alpha)]] = 0$, it is directly apparent that this yields another classical dynamical r-matrix for $(\mathfrak{h}, \mathfrak{iso}(2,1))$ and only modifies r by adding a twist.

Note in particular that $r_{a,b}$ are invariant under gauge transformations $\Pi : \mathfrak{h}_{a,b} \to H$ of the form $\Pi(\psi, \alpha) = (\exp(-\beta(\psi)J_j), -\alpha\beta'(\psi)P_j)$ and $\Pi(\psi, \alpha) = (\mathbb{1}, -\gamma(\psi)P_j)$. The former correspond to combinations of a Lorentz transformation that preserves $\mathfrak{h}_{a,b}$ and a translation in the direction of its axis. The latter correspond to translations which do not depend on the parameter α. By specialising formula (22) to the case at hand in which $\boldsymbol{q}_\psi = \boldsymbol{q}_\alpha = e_j$, $\boldsymbol{q}_\theta = 0$ one finds that these are precisely the flows that the variables $\psi \cdot \alpha$ and ψ generate via the bracket $\{\,,\,\}_D$ in Definition 4.6.

5.2 Standard solutions and classical dynamical r-matrices

Although Theorem 4.8 provides a direct link between the Jacobi identity for the bracket in Definition 4.6 and solutions of the CDYBE that are subject to the additional conditions (25), the disadvantage of this description is that the associated solutions $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ are in general quite complicated and the additional conditions (25) do not have an immediate geometrical interpretation.

It is therefore natural to ask if they can be related to a simple set of standard solutions which define classical dynamical r-matrices in the sense of Definition 4.7. The results of the previous subsection suggest that this can be achieved by applying dynamical Poincaré transformations. As we will see in the following, this is possible for those values of the parameters ψ for which $\boldsymbol{q}_\psi$ and $\boldsymbol{q}_\alpha$ are timelike or spacelike. A necessary and sufficient condition then is that the maps $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ and $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta : \mathbb{R}^2 \to \mathbb{R}^3$ satisfy the equations (25) in Theorem 4.8. We have the following lemma:

Lemma 5.5. *Consider $r : I \times \mathbb{R} \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ as in* (23) *and $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta : I \times \mathbb{R} \to \mathbb{R}^3$, where $I \subset \mathbb{R}$ is an open interval with $\boldsymbol{q}_\psi^2(\psi), \boldsymbol{q}_\alpha^2(\psi) \neq 0$, $\boldsymbol{q}_\psi(\psi) \wedge \boldsymbol{q}_\alpha(\psi) = 0$ and $\partial_\alpha \boldsymbol{q}_\psi(\psi) = \partial_\alpha \boldsymbol{q}_\alpha(\psi) = \partial_\alpha^2 \boldsymbol{q}_\theta(\psi) = 0$ for all $\psi \in I$. If r and $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta$ satisfy the conditions* (25)*, then there exists a Poincaré transformation as in Lemma 5.1 such that the transformed quantities $r^p : I \times \mathbb{R} \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$, $\boldsymbol{q}^p_\psi, \boldsymbol{q}^p_\alpha, \boldsymbol{q}^p_\theta : I \times \mathbb{R} \to \mathbb{R}^3$ defined by* (32) *are of the form*

$$r^p = \tfrac{1}{2}(P_a \otimes J^a + J^a \otimes P_a) - \tfrac{1}{2}\varepsilon^{abc} w^p_c P_a \wedge J_b + \tfrac{1}{2}\varepsilon^{abc} m^p_c P_a \wedge P_b, \tag{37}$$

with one of the following:

a) $\boldsymbol{q}^p_\psi, \boldsymbol{q}^p_\alpha, \boldsymbol{q}^p_\theta, \boldsymbol{w}^p, \boldsymbol{m}^p \in \text{span}\{\boldsymbol{e}_0\}$ *and* $q^{p,0}_\alpha \partial_\alpha q^{p,0}_\theta = q^{p,0}_\psi \partial_\psi q^{p,0}_\alpha$,

b) $\boldsymbol{q}_\psi^p, \boldsymbol{q}_\alpha^p, \boldsymbol{q}_\theta^p, \boldsymbol{w}^p, \boldsymbol{m}^p \in \mathrm{span}\{\boldsymbol{e}_1\}$ *and* $q_\alpha^{p,1}\partial_\alpha q_\theta^{p,1} = q_\psi^{p,1}\partial_\psi q_\alpha^{p,1}$.

Proof.

1. Let r be of the form (23) and $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta : I \times \mathbb{R} \to \mathbb{R}^3$ with $\boldsymbol{q}_\psi \wedge \boldsymbol{q}_\alpha = 0$ and either $\boldsymbol{q}_\psi^2, \boldsymbol{q}_\alpha^2 > 0$ (case a) or $\boldsymbol{q}_\psi^2, \boldsymbol{q}_\alpha^2 < 0$ (case b). Then the formulas for the action of dynamical Lorentz transformations in Lemma 5.1 imply that via a suitable Lorentz transformation $g : \mathbb{R}^2 \to \mathrm{SO}_+(2,1)$, $\partial_\alpha g = 0$, we can achieve one of the following: a) $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha \in \mathrm{span}\{\boldsymbol{e}_0\}$ or b) $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha \in \mathrm{span}\{\boldsymbol{e}_1\}$.

   The resulting matrix $V : \mathbb{R}^2 \to \mathrm{Mat}(3,\mathbb{R})$ in (23) can be decomposed into a symmetric and an antisymmetric component according to

   $$V^{ab}(\psi) = \tfrac{1}{2}\eta^{ab} + Q^{ab}(\psi) + \tfrac{1}{2}\varepsilon^{abc}w_c(\psi)$$

   with $\boldsymbol{w} : I \to \mathbb{R}^3$ and $Q^{ab} : I \to \mathrm{Mat}(3,\mathbb{R})$ symmetric. By applying a suitable translation $\boldsymbol{t} : I \times \mathbb{R} \to \mathbb{R}^3$, $\partial_\alpha^2 \boldsymbol{t} = 0$, which does not affect $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha$, one can then achieve that the symmetric matrix $Q^{ab} : I \times \mathbb{R} \to \mathrm{Mat}(3,\mathbb{R})$ satisfies $Q^{0a} = Q^{a0} = 0$ $\forall a \in \{0,1,2\}$ in case a) and $Q^{1a} = Q^{a1} = 0$ $\forall a \in \{0,1,2\}$ in case b). With a further translation $\boldsymbol{t}' : \mathbb{R}^2 \to \mathbb{R}^3$ which satisfies $\partial_\alpha \boldsymbol{t}' = 0$ and hence does not affect V, one can achieve that $\boldsymbol{q}_\theta$ takes the form $\boldsymbol{q}_\theta = \alpha \cdot \partial_\alpha \boldsymbol{q}_\theta + \tilde{\boldsymbol{q}}_\theta$ with $\partial_\alpha \tilde{\boldsymbol{q}}_\theta = 0$ and $\tilde{\boldsymbol{q}}_\theta \in \mathrm{span}\{\boldsymbol{e}_0\}$ in case a) or $\tilde{\boldsymbol{q}}_\theta \in \mathrm{span}\{\boldsymbol{e}_1\}$ in case b).

2. After these transformations, the first condition in (25) is satisfied if and only if $\boldsymbol{w} \in \mathrm{span}\{\boldsymbol{e}_0\}$ and $Q^{11} + Q^{22} = 0$ in case a) and $\boldsymbol{w} \in \mathrm{span}\{\boldsymbol{e}_1\}$ and $Q^{00} - Q^{22} = 0$ in case b). Under these conditions, the second equation in (25) simplifies to

   $$q_\alpha^a \partial_\alpha q_\theta^b - q_\psi^b \partial_\psi q_\alpha^a + q_\alpha^d(\varepsilon^a{}_{dh}Q^{bh} + \varepsilon^b{}_{dh}Q^{ah}) = 0 \qquad \forall a,b \in \{0,1,2\}.$$

   This implies $\partial_\alpha \boldsymbol{q}_\theta \in \mathrm{span}\{\boldsymbol{e}_0\}$, $Q = 0$ in case a) and $\partial_\alpha \boldsymbol{q}_\theta \in \mathrm{span}\{\boldsymbol{e}_1\}$, $Q = 0$ in case b). Combining this with the previous results, we obtain $\boldsymbol{q}_\theta \in \mathrm{span}\{\boldsymbol{e}_0\}$ and $q_\alpha^0 \partial_\alpha q_\theta^0 = q_\psi^0 \partial_\psi q_\alpha^0$ in case a) and $\boldsymbol{q}_\theta \in \mathrm{span}\{\boldsymbol{e}_1\}$ and $q_\alpha^1 \partial_\alpha q_\theta^1 = q_\psi^1 \partial_\psi q_\alpha^1$ in case b). Inserting these conditions into the third equation in (25), one finds that this equation simplifies to the condition $\boldsymbol{m} \wedge \boldsymbol{q}_\alpha = 0$, which implies $\boldsymbol{m} \in \mathrm{span}\{\boldsymbol{e}_0\}$ in case a) and $\boldsymbol{m} \in \mathrm{span}\{\boldsymbol{e}_1\}$ in case b). This proves the claim.

□

If $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ and $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta : \mathbb{R}^2 \to \mathbb{R}^3$ are of the form in Lemma 5.5, then $\mathfrak{h}(\psi,\alpha) = \mathrm{span}\{q_\psi^a P_a, q_\alpha^a J_a + q_\theta^a P_a\}$ is a fixed Cartan subalgebra of $\mathfrak{iso}(2,1)$ which no longer varies with ψ and α. Moreover, a direct calculation shows that $r(\psi,\alpha)$ is then invariant under the action of the Cartan subalgebra $\mathfrak{h}(\psi,\alpha)$: $[\boldsymbol{y} \otimes 1 + 1 \otimes \boldsymbol{y}, r(\psi,\alpha)] = 0$ for all $\boldsymbol{y} \in \mathfrak{h}(\psi,\alpha)$.

This provides us with a natural geometrical interpretation of the conditions (25) in Theorem 4.8. These conditions ensure that for all values of ψ for which $\boldsymbol{q}_\psi^2(\psi), \boldsymbol{q}_\alpha^2(\psi) \neq 0$, the maps $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$, $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta : \mathbb{R}^2 \to \mathbb{R}^3$ (which are not required to satisfy the CDYBE in Lemma 5.5) can be brought into a standard form via a suitable dynamical Poincaré transformation. The resulting map $r : \mathbb{R}^2 \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ is then invariant under the fixed Cartan subalgebra spanned by $\boldsymbol{q}_\psi, \boldsymbol{q}_\theta, \boldsymbol{q}_\alpha$. The conditions (25) can therefore be viewed as a generalised or Poincaré-transformed version of the restriction to a fixed Cartan

subalgebra in Definition 4.7. As shown in the proof of Theorem 4.8, they ensure that the Jacobi identity holds for mixed brackets involving functions of the variables ψ, α and functions on $\mathrm{ISO}(2,1)^{n-2+2g}$.

Lemma 5.5 applies in particular to solutions of the CDYBE that satisfy the conditions (25) and hence give rise to Poisson structures $\{\,,\}_D$ on $\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$. This allows one to (locally) classify all possible solutions of the CDYBE that arise from gauge fixing conditions satisfying the conditions in Section 3.5 and hence all associated Dirac brackets.

Theorem 5.6. *Let $r : I \times \mathbb{R} \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ be a solution of the CDYBE with $x^1 = \psi$, $x^2 = \alpha$, $x_1 = q^a_\psi P_a$, $x_2 = q^a_\alpha J_a + q^a_\theta P_a$ that satisfies conditions (25) in Theorem 4.8 and for which $\boldsymbol{q}_\psi^2, \boldsymbol{q}_\alpha^2 \neq 0$, $\boldsymbol{q}_\psi \wedge \boldsymbol{q}_\alpha = 0$ and $\partial_\alpha \boldsymbol{q}_\psi = \partial_\alpha \boldsymbol{q}_\alpha = \partial_\alpha^2 \boldsymbol{q}_\theta = 0$ on $I \times \mathbb{R}$. Then there exists a Poincaré transformation $p : I \times \mathbb{R} \to \mathrm{ISO}(2,1)$ as in Lemma 5.5 and a diffeomorphism $\boldsymbol{y} = (y^1, y^2) : I \times \mathbb{R} \to I' \times \mathbb{R}$ with $\partial_\alpha y_1 = 0$ and $\partial_\alpha^2 y_2 = 0$ such that one of the following holds:*

a) $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta \in \mathrm{span}\{\boldsymbol{e}_0\}$ *and*

$$r^p(\psi,\alpha) = \tfrac{1}{2}(P_a \otimes J^a + J^a \otimes P_a) + \tfrac{1}{2}\tan\tfrac{y^1(\psi)}{2}\,(P_1 \wedge J_2 - P_2 \wedge J_1) + \frac{y^2(\psi,\alpha)}{4\cos^2\frac{y^1(\psi)}{2}} P_1 \wedge P_2,$$

b) $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta \in \mathrm{span}\{\boldsymbol{e}_1\}$ *and*

$$r^p(\psi,\alpha) = \tfrac{1}{2}(P_a \otimes J^a + J^a \otimes P_a) + \tfrac{1}{2}\tanh\tfrac{y^1(\psi)}{2}\,(P_2 \wedge J_0 - P_0 \wedge J_2) + \frac{y^2(\psi,\alpha)}{4\cosh^2\frac{y^1(\psi)}{2}} P_2 \wedge P_0.$$

Proof.

1. By Lemma 5.5 there exists a Poincaré-valued function $p : I \times \mathbb{R} \to \mathrm{ISO}(2,1)$ whose Lorentzian component does not depend on α and whose translation component depends on α at most linearly such that $r^p : I \times \mathbb{R} \to \mathfrak{iso}(2,1) \otimes \mathfrak{iso}(2,1)$ and $\boldsymbol{q}^p_\psi, \boldsymbol{q}^p_\alpha, \boldsymbol{q}^p_\theta : I \times \mathbb{R} \to \mathbb{R}^3$ satisfy again the CDYBE as well as conditions (25) and are of the form given in Lemma 5.5. This implies that r is of the form (37):

$$r^p = \tfrac{1}{2}(P_a \otimes J^a + J^a \otimes P_a) - \tfrac{1}{2}\varepsilon^{abc} w^p_c P_a \wedge J_b + \tfrac{1}{2}\varepsilon^{abc} m^p_c P_a \wedge P_b$$

with $\boldsymbol{q}^p_\psi, \boldsymbol{q}^p_\alpha, \boldsymbol{q}^p_\theta, \boldsymbol{w}^p, \boldsymbol{m}^p$ subject to conditions a) or b) in Lemma 5.5. It follows that there are smooth functions $\beta, \gamma, \delta, \epsilon, \varphi_0, \varphi_1 : I \to \mathbb{R}$ with $\beta(\psi), \gamma(\psi) \neq 0$ for all $\psi \in I$ such that

$$\boldsymbol{q}^p_\psi = \beta \boldsymbol{e}_j, \quad \boldsymbol{q}^p_\alpha = \gamma \boldsymbol{e}_j, \quad \boldsymbol{q}^p_\theta = (\delta + \alpha\,\gamma'\beta/\gamma)\,\boldsymbol{e}_j, \quad \boldsymbol{w}^p = \epsilon\,\boldsymbol{e}_j, \quad \boldsymbol{m}^p = (\varphi_0 + \alpha\varphi_1)\boldsymbol{e}_j,$$

where $j = 0$ in case a) and $j = 1$ in case b). Inserting these expressions into (29), one finds that r^p satisfies the CDYBE with $x^1 = \psi$, $x^2 = \alpha$, $x_1 = q^{p,a}_\psi P_a$ and $x_2 = q^{p,a}_\alpha J_a + q^{p,a}_\theta P_a$ if and only if the coefficient functions $\beta, \gamma, \delta, \epsilon, \varphi_0, \varphi_1 : I \to \mathbb{R}$ satisfy the following set of differential equations:

$$\gamma\varphi_1 + \tfrac{1}{2}\beta\epsilon' = 0, \qquad 1 \pm \epsilon^2 \pm 2\beta\epsilon' = 0, \qquad \beta\varphi_0' + \delta\varphi_1 + \epsilon\varphi_0 = 0, \tag{38}$$

where the sign $+$ in the second equation refers to case a), the sign $-$ to case b). Set $g = 1/\gamma$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function with $f' = 1/\beta$. Then the second equation can be integrated to $\epsilon(\psi) = -\tan(f(\psi)/2)$ in case a) and $\epsilon(\psi) = \tanh(f(\psi)/2)$ in case b). Inserting this result into the remaining two equations, we obtain

$$\varphi_1(\psi) = -\tfrac{1}{2} g(\psi)\epsilon'(\psi), \qquad \delta(\psi) = -\frac{h'(\psi)}{f'(\psi)g(\psi)},$$

with $h(\psi) = 4\varphi_0(\psi)\cos^2(f(\psi)/2)$ in case a) and $h(\psi) = 4\varphi_0(\psi)\cosh^2(f(\psi)/2)$ in case b). It follows that $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha, \boldsymbol{q}_\theta, \boldsymbol{w}, \boldsymbol{m}$ are given by:

$$\left.\begin{aligned}
&\boldsymbol{q}^p_\psi(\psi) = \frac{\boldsymbol{e}_0}{f'(\psi)}, \quad \boldsymbol{q}^p_\alpha(\psi) = \frac{\boldsymbol{e}_0}{g(\psi)}, \quad \boldsymbol{q}^p_\theta(\psi,\alpha) = -\frac{h'(\psi)+\alpha g'(\psi)}{f'(\psi)g(\psi)}\,\boldsymbol{e}_0,\\
&\boldsymbol{w}^p(\psi) = -\tan(f(\psi)/2)\,\boldsymbol{e}_0, \quad \boldsymbol{m}^p(\psi,\alpha) = \frac{h(\psi)+\alpha g(\psi)}{4\cos^2(f(\psi)/2)}\,\boldsymbol{e}_0 \quad \text{in case a)},\\
&\boldsymbol{w}^p(\psi) = \tanh(f(\psi)/2)\,\boldsymbol{e}_0, \quad \boldsymbol{m}^p(\psi,\alpha) = -\frac{h(\psi)+\alpha g(\psi)}{4\cosh^2(f(\psi)/2)}\,\boldsymbol{e}_0 \quad \text{in case b)}.
\end{aligned}\right\} \tag{39}$$

2. With the definitions $y^1(\psi) = f(\psi)$, $y^2(\psi,\alpha) = h(\psi) + \alpha g(\psi)$ this yields expressions a) and b) for r. For any $y_1, y_2 \in \mathfrak{iso}(2,1)$ the right-hand side of the CDYBE then takes the form

$$\begin{aligned}
&y_1^{(1)}\partial_{y^1} r_{23} - y_1^{(2)}\partial_{y^1} r_{13} + y_1^{(3)}\partial_{y^1} r_{12} + y_2^{(1)}\partial_{y^2} r_{23} - y_2^{(2)}\partial_{y^2} r_{13} + y_2^{(3)}\partial_{y^2} r_{12}\\
&= x_1^{(1)}\partial_\psi r_{23} - x_1^{(2)}\partial_\psi r_{13} + x_1^{(3)}\partial_\psi r_{12} + x_2^{(1)}\partial_\alpha r_{23} - x_2^{(2)}\partial_\alpha r_{13} + x_2^{(3)}\partial_\alpha r_{12},
\end{aligned}$$

where

$$x_1(\psi) = \frac{y_1(\psi)}{f'(\psi)}, \qquad x_2(\psi,\alpha) = \frac{y_2(\psi,\alpha)}{g(\psi)} - \frac{(h'(\psi)+\alpha g'(\psi))\,y_1(\psi)}{f'(\psi)g(\psi)}.$$

Setting $y_1 = P_0$, $y_2 = J_0$ in case a) and $y_1 = P_1$, $y_2 = J_1$ in case b), we obtain $x_1 = q_\psi^{p,a} P_a$, $x_2 = q_\alpha^{p,a} J_a + q_\theta^{p,a} P_a$ with $\boldsymbol{q}^p_\psi, \boldsymbol{q}^p_\alpha, \boldsymbol{q}^p_\theta$ given by (39). This proves the claim.

□

Theorem 5.6 amounts to a classification of all possible solutions of the CDYBE (24) of the form in Definition 4.6 that satisfy the additional conditions (25) and hence of all Poisson structures of the type in Definition 4.6. It states that locally all such Poisson structures are obtained from one of the classical dynamical r-matrices in Lemma 4.11 by applying a suitable Poincaré transformation together with a rescaling of the variables ψ, α. Note, however, that this classification is only local in the following sense: for any given value ψ_0 of the variable ψ for which $\boldsymbol{q}^2_\psi, \boldsymbol{q}^2_\alpha \neq 0$, there is an open interval $I \subset \mathbb{R}$, $\psi_0 \in I$, such that $r(\psi,\alpha)$ can be transformed into one of the classical dynamical r-matrices in Theorem 5.6 for all $(\psi,\alpha) \in I \times \mathbb{R}$.

In particular, this locally determines all possible Dirac brackets that arise from gauge fixing procedures that satisfy the well-motivated conditions in Section 3.5. The Dirac bracket of such a gauge fixing procedure is always determined by a solution of the CDYBE that satisfies the additional conditions (25) and $\boldsymbol{q}_\psi \wedge \boldsymbol{q}_\alpha = 0$. For those values of the variable ψ for which $\boldsymbol{q}^2_\psi(\psi), \boldsymbol{q}^2_\alpha(\psi) \neq 0$, the resulting Dirac bracket can be transformed into the bracket defined by one of the two classical dynamical r-matrices in Lemma 4.11.

However, the Dirac bracket resulting from a generic gauge fixing condition is associated with maps $\boldsymbol{q}_\psi, \boldsymbol{q}_\alpha : \mathbb{R} \to \mathbb{R}^3$ for which the signature of $\boldsymbol{q}^2_\psi, \boldsymbol{q}^2_\alpha$ changes as a function of the variable ψ. In particular, this is the case for the specific gauge fixing conditions investigated in [28]. It is shown there that the map $\boldsymbol{q}_\psi : \mathbb{R}^2 \to \mathbb{R}^3$ is timelike, lightlike or spacelike for those values of ψ for which, respectively, the product $u_{M_2} \cdot u_{M_1}$ of the Lorentzian components of the gauge-fixed holonomies is elliptic, parabolic or hyperbolic. It is well-known that the product

of two elliptic elements of $SO_+(2,1)$ can be either elliptic, parabolic or hyperbolic. The first requirement on the gauge fixing conditions in Section 3.4 implies that all of these cases must arise in the Dirac bracket. This suggests that such transitions between timelike, spacelike and lightlike solutions are a generic outcome of the gauge fixing procedure when the two holonomies M_1, M_2 are elliptic.

It is therefore not possible to reduce the investigation of gauge fixing procedures and the resulting Poisson structures to classical dynamical r-matrices in the sense of Definition 4.7. Instead, one needs to admit more general solutions of the CDYBE and to allow the associated Lie subalgebras $\mathfrak{h}(\psi, \alpha) \subset \mathfrak{iso}(2,1)$ to vary non-trivially with the variables ψ and α. Such solutions are no longer invariant under the action of the subalgebra $\mathfrak{h}(\psi, \alpha)$. Instead, they satisfy the generalised consistency conditions (25) which together with the CDYBE ensure the Jacobi identity for the associated Poisson bracket.

6 Outlook and conclusions

In this paper we applied the Dirac gauge fixing procedure to the description of the moduli space of flat $ISO(2,1)$-connections in terms of an ambient space with an auxiliary Poisson structure [22]. We investigated a large class of gauge fixing conditions subject to two well-motivated structural requirements, namely that the gauge fixing is based on the choice of two punctures on the underlying Riemann surface and that it preserves the natural $\mathbb{N}$-grading of the auxiliary Poisson structure.

We showed that the Poisson algebras obtained from gauge fixing are in one-to-one correspondence with solutions of the classical dynamical Yang-Baxter equation (CDYBE) and of a set of additional equations. The latter can be viewed as the counterpart of the usual requirement of invariance of the classical dynamical r-matrices under the action of a fixed Cartan subalgebra. These solutions of the CDYBE are a generalisation of classical dynamical r-matrices in which the associated two-dimensional Lie subalgebras of $\mathfrak{iso}(2,1)$ vary with the dynamical variables.

We also demonstrated how a change of gauge fixing conditions affects the associated solutions of the CDYBE and showed that this change corresponds to the action of dynamical Poincaré transformations. These dynamical Poincaré transformations generalise the gauge transformations of classical dynamical r-matrices in [19]. We found that every solution obtained via gauge fixing can be transformed into one of two classical dynamical r-matrices via these transformations and a rescaling for almost all values of the dynamical variables. This gives rise to a complete (local) classification of all possible outcomes of gauge fixing in the context of the moduli space of flat $ISO(2,1)$-connections.

However, for generic solutions there are also certain values of the dynamical variables for which the solutions cannot be transformed into standard classical dynamical r-matrices. These singular points appear at the transition between classical dynamical r-matrices for two non-conjugate Cartan subalgebras of $\mathfrak{iso}(2,1)$ and are associated with two-dimensional Lie subalgebras of $\mathfrak{iso}(2,1)$ which contain parabolic elements of $\mathfrak{so}(2,1)$. This occurs for instance in the solutions in Lemma 4.9 and cannot be excluded by suitable gauge fixing conditions.

To our knowledge this phenomenon does not appear in earlier references on the topic. We expect that this is due to the fact that most of these references investigate classical dynamical r-matrices for complex (semi)simple Lie groups, for which all Cartan subalgebras are conjugate. Based on our results, we would predict that such transitions between classical dynamical

r-matrices for non-conjugate Cartan subalgebras arise for real Lie groups for which the underlying symmetric, non-degenerate Ad-invariant symmetric form $\langle\, ,\, \rangle$ is indefinite.

Another aspect which merits further investigation is the physical interpretation of these classical dynamical r-matrices in the context of (2+1)-dimensional (quantum) gravity. It was shown in [28] that gauge fixing procedures of the type investigated in this paper can be viewed as the specification of an observer in a (2+1)-dimensional spacetime. Moreover, the results in [28] suggest a direct geometrical interpretation for the two dynamical variables in the classical dynamical r-matrices: they correspond to the total mass and angular momentum of the spacetime as measured by this observer. In this interpretation, the two standard classical dynamical r-matrices would be associated with the centre-of-mass frame of the universe and the transition points between different Cartan subalgebras would correspond to the formation of Gott pairs [23].

We expect that our results could be generalised to the other moduli spaces of flat connections that arise in the description of (2+1)-gravity for different signatures and different values of the cosmological constant. For Lorentzian signature, the relevant Lie groups are $\mathrm{ISO}(2,1)$, $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$, respectively, for vanishing, negative and positive cosmological constant. For Euclidean signature, the corresponding Lie groups are $\mathrm{ISO}(3)$, $\mathrm{SO}(3) \times \mathrm{SO}(3)$ and $\mathrm{PSL}(2,\mathbb{C})$. It seems plausible that analogous gauge fixing procedures applied to these groups would yield similar outcomes, with the exception of the transition between different Cartan subalgebras, which should not occur for Euclidean signature.

In this context, it would also be desirable to understand in more detail how our results are related to the classical dynamical r-matrix symmetries obtained by Buffenoir, Noui and Roche via a regularisation procedure for point particles coupled to Chern-Simons theory with gauge group $\mathrm{SL}(2,\mathbb{C})$ [10, 13, 32]. Although the approach and setting in this work are very different, there should be an underlying reason that forces the appearance of classical dynamical r-matrices in both cases.

It would also be interesting to investigate in more detail the relation between gauge-fixed Poisson structures on the moduli space of flat $\mathrm{ISO}(2,1)$-connections and mathematical structures associated with classical dynamical r-matrices such as Poisson-Lie groupoids. As the auxiliary Poisson algebra which is gauge-fixed is closely related to certain structures from the theory of Poisson-Lie groups, namely the dual Poisson-Lie structure and the Heisenberg double, it could be expected that gauge fixing should be related to the construction of dynamical versions of these structures.

Finally, we expect our results to have useful application in the quantisation of the moduli space of flat $\mathrm{ISO}(2,1)$-connections. This is due to the fact that the resulting Poisson structure is very closely related to Fock and Rosly's Poisson structure on the ambient space. The only difference is that the classical r-matrix is replaced by a classical dynamical r-matrix associated with two-dimensional Lie subalgebras of $\mathfrak{iso}(2,1)$. As Fock and Rosly's Poisson structure serves as the starting point for the combinatorial quantisation formalism, this suggests that this formalism could be extended straightforwardly to include the gauge-fixed Poisson structure. This would reduce the task of quantising the theory to the construction of the associated dynamical quantum group.

Acknowledgements

This work was supported by the Emmy Noether research grant ME 3425/1-1 of the German Research Foundation (DFG). We thank Karim Noui for useful remarks and discussions and Stefan Waldmann and Winston Fairbairn for comments on a draft of this paper. The xAct tensor calculus package [25] proved useful in some of the computations.

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